Secondary Mathematics 2 Table of Contents

Unit 4

Applications of Probability

Unit 4 Cluster 1 (S.CP.1)
Applications of Probability

Cluster 1: Understand independence and conditional probability and use them to interpret data

4.1.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

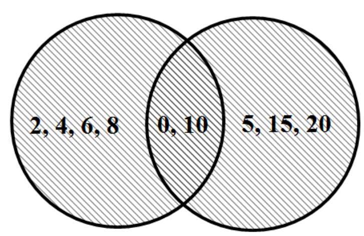
VOCABULARY

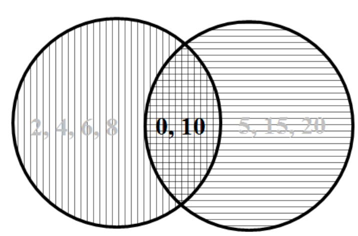
An **event** is an activity or experiment which is usually represented by a capital letter. A **sample space** is a set of all possible outcomes for an activity or experiment. A smaller set of outcomes from the sample space is called a **subset**. The **complement** of a subset is all outcomes in the sample space that are not part of the subset. A subset and its complement make up the entire sample space. If a subset is represented by A, the complement can be represented by any of the following: not A, ~A, or A^c.

Example 1:

Event	Sample Space	Possible Subset	Complement
Flip a coin	S={heads, tails}	B={heads}	$B^c = \{\text{tails}\}$
Roll a die	S={1, 2, 3, 4, 5, 6}	even E={2, 4, 6}	~E ={1, 3, 5}
Pick a digit 0-9	S={0,1, 2, 3, 4, 5, 6, 7, 8, 9}	N={2, 5, 7, 9}	not N= {0, 1, 3, 4, 6, 8}

<table>
<tr><th colspan="3">VOCABULARY</th></tr>
<tr><th>Definition</th><th>Example</th><th>Venn Diagram</th></tr>
<tr><td>The union of two events includes all outcomes from each event. The union can be indicated by the word "or" or the symbol $\cup$.</td><td>$A=\{0,2,4,6,8,10\}$
$B=\{0,5,10,15,20\}$

$A\cup B=\{0,2,4,5,6,8,10,15,20\}$</td><td>
</td></tr>
<tr><td>The intersection of two events includes only those outcomes that are in both events. The intersection can be indicated by the word "and" or the symbol $\cap$. If the two events do NOT have anything in common, the intersection is the "empty set", indicated by { } or $\varnothing$.</td><td>$A=\{0,2,4,6,8,10\}$
$B=\{0,5,10,15,20\}$

$A\cap B=\{0,10\}$</td><td>
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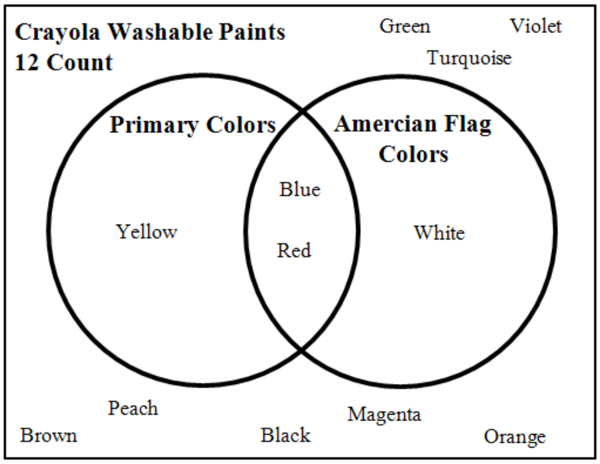
Example 2:

	**The sample space is S={Green, Violet, Turquoise, Yellow, Blue, Red, White, Brown, Peach, Black, Magenta, Orange}** **The subset of primary colors is P={Yellow, Blue, Red}** **The subset of American Flag colors is: A={Blue, Red, White}** $P \cup A$= **{Yellow, Blue, Red, White}** $P \cap A$= **{Blue, Red}** P^c = **{Green, Violet, Turquoise, White, Brown, Peach, Black, Magenta, Orange}** $\sim(P \cup A)$ **={Green, Violet, Turquoise, Brown, Peach, Black, Magenta, Orange}**

Practice Exercises A

1. Choosing a letter from the alphabet.
   A. List the sample space.
   B. List a subset of the letters in your first name.
   C. List a subset of the letters in your last name.
   D. Find the union of the subsets of your first name and last name.
   E. Find the intersection of the subsets of your first name and last name.

2. Given the sample space S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} with event A = {3, 4, 5, 6, 7} and event B = {1, 2, 3, 4, 5}.
   A. Draw a Venn diagram representing the sample space with events A and B.
   B. List all the outcomes for $A \cup B$.
   C. List all the outcomes for $A \cap B$.
   D. List all the outcomes for A^c.

3. Given a standard deck of 52 cards, event A is defined as a red card and event B is defined as the card is a diamond.
   A. List all the outcomes for $A \cup B$.
   B. List all the outcomes for $A \cap B$.
   C. What is ~A?
   D. What is $A \cap A^c$?

Unit 4 Clusters 1–2 (S.CP.2, S.CP.3, S.CP.4, S.CP.5, S.CP.6, S.CP.7) Conditional Probability

Cluster 1: Understanding and using independence and conditional probability

- 4.1.2 Independence of 2 events (use the product of their probabilities to determine if they are independent)
- 4.1.3 Understand conditional probability and interpret the independence of A and B using conditional probability
- 4.1.4 Construct and interpret two-way frequency tables; Use two-way frequency tables to determine independence and to find conditional probabilities
- 4.1.5 Explain conditional probability and independence

Cluster 2: Computing probabilities of compound events

- 4.2.1 Find conditional probabilities
- 4.2.2 Apply the Addition Rule

VOCABULARY

Probability is a value that represents the likelihood that an event will occur. It can be represented as a fraction, decimal $\left(0 \le probabiltiy \le 1\right)$, or percent. A probability of zero (0) means that the event is impossible and a probability of one (1) means that the event must occur.

Events are **independent** if the occurrence of one event does not change the probability of another event occurring. Events are **dependent** if the occurrence of one changes the probability of another event occurring. For example, drawing marbles from a bag with replacement is independent, while drawing marbles from a bag without replacement is dependent.

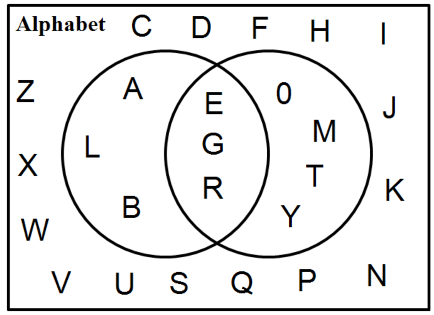
Joint probability is the likelihood of two or more events occurring at the same time.

Formula	**Description**	**Example**
$P\left(A\right)=\dfrac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$	Probability of the individual event A occurring.	Flipping a coin P(heads)= $\dfrac{1}{2}$
$P(A \cap B)=P(A)\cdot P(B)$	Joint probability of independent events.	Flipping a coin AND rolling a die $P(\text{heads and }5)=P(\text{heads})\cdot P(5)$ $=\dfrac{1}{2}\cdot\dfrac{1}{6}$ $=\dfrac{1}{12}$

The Addition Rule	
$P(\mathrm{A}\cup\mathrm{B})=P(\mathrm{A})+P(\mathrm{B})-P(\mathrm{A}\cap\mathrm{B})$	The addition rule finds the probability of event A occurring or event B occurring.

A letter in the word ***Algebra*** or a letter in the word ***Geometry***. Where event A is Algebra and event B is Geometry.

$$P(\mathrm{A}\cup\mathrm{B})=P(\mathrm{A})+P(\mathrm{B})-P(\mathrm{A}\cap\mathrm{B})$$
$$=\frac{6}{26}+\frac{7}{26}-\frac{3}{26}$$
$$=\frac{10}{26}=\frac{5}{13}$$

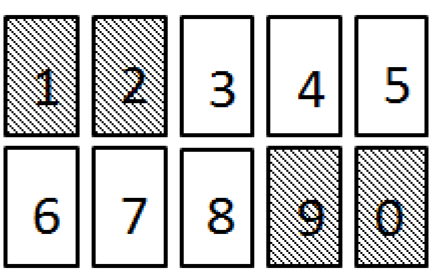


Practice Exercises A

1. You have an equally likely chance of choosing an integer from 1 to 50. Find the probability of each of the following events.
   A. An even number
   B. A perfect square
   C. A factor of 150 is chosen
   D. A two digit number is chosen
   E. A multiple of 4 is chosen
   F. A number less than 35 is chosen
   G. A prime number is chosen
   H. A perfect cube is chosen

2. You randomly chose two marbles, replacing the first marble before drawing again, from a bag containing 10 black, 8 red, 4 white, and 6 blue marbles. Find the probability of each of the following events.
   A. A white marble, then a red marble is selected.
   B. A red marble is not selected, then a blue marble is selected.
   C. A green marble, then a green marble is selected.
   D. A blue or black marble is selected, then a white marble is selected.

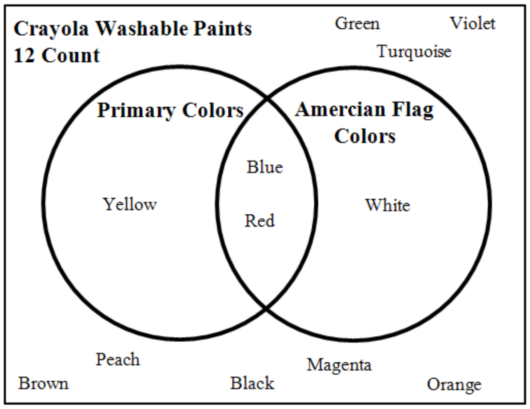
3.



Drawing a card from the cards on the left, determine the probability of each of the following.

A. P(Even or shaded)
B. P(White or odd)
C. P(Less than four or shaded)
D. P(Greater than five or shaded)
E. P(Factor of ten or white)

Example 1:



P(peach)= $\frac{1}{12}$

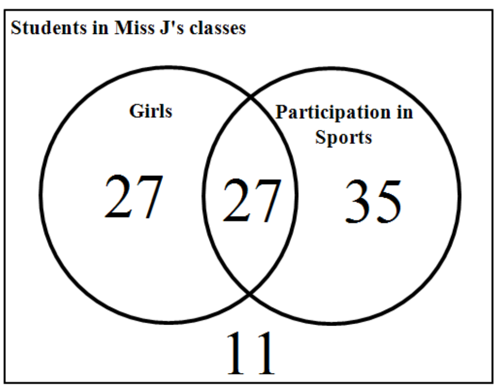
P(color in American flag)= $\frac{3}{12}=\frac{1}{4}$

P(primary color and American flag)= $\frac{2}{12}=\frac{1}{6}$

P(pink)= $\frac{0}{12}=0$

Practice Exercises B

Using the Venn diagram, answer the following questions.



1. P(girls)
2. P(sports, not girls)
3. P(not sports)
4. P(not sports, not girls)
5. P(girls and sports)
6. P(Mr. P class)

Example 2:

	Curfew: Yes	**Curfew: No**	**Total**
Chores: Yes	**13**	**5**	**18**
Chores: No	**12**	**3**	**15**
Total	**25**	**8**	**33**

P(has chores)= $\frac{18}{33}$

P(doesn't have a curfew)= $\frac{8}{33}$

P(has a curfew and chores)= $\frac{13}{33}$

P(has chores $\cap$ doesn't have a curfew)= $\frac{5}{33}$

P(has a curfew)= $\frac{25}{33}$

Practice Exercises C

Find the marginal totals. Then use the table to find the probabilities below.

	Brown hair	Blonde hair	Red hair	Black hair	Other hair	Total
Male	42	11	3	17	27	
Female	47	16	13	9	15	
Total						

1. P(male)
2. P(red hair)
3. P(other hair)
4. P(blonde hair $\cap$ male)
5. P(black hair and female)
6. P(brown hair $\cap$ not male)
7. P(female $\cap$ not other hair)
8. P(not female $\cap$ not male)
9. P(red hair and black hair)

VOCABULARY

Two events are **independent** if $P(A)\cdot P(B)=P(A\cap B)$

Example 3:

	10^{th}	11^{th}	12^{th}	Total
Male	320	297	215	832
Female	285	238	216	739
Total	605	535	431	1571

1. Are being a male and being in 10^{th} independent?

2. Are being a female and being in 12^{th} grade independent?

1.

$P(\text{male})=\frac{832}{1571}$

$P(10^{th}\text{ grade})=\frac{605}{1571}$

$P(\text{male }\cap 10^{th}\text{ grade})=\frac{320}{1571}$

$$\frac{832}{1571}\cdot\frac{605}{1571}\overset{?}{=}\frac{320}{1571}$$

$$0.204=0.204$$

The product of the probabilities of the individual events is equal to the probability of the intersection of the events; therefore the events are independent.

2.

$P(\text{female})=\frac{739}{1571}$

$P(12^{th}\text{ grade})=\frac{431}{1571}$

$P(\text{female }\cap 12^{th}\text{ grade})=\frac{216}{1571}$

$$\frac{739}{1571}\cdot\frac{431}{1571}\overset{?}{=}\frac{216}{1571}$$

$$0.129\neq 0.137$$

The product of the probabilities of the individual events is not equal to the probability of the intersection of the events; therefore the events are not independent.

Practice Exercises D
Determine whether or not the following events are independent.

1. If P(A)=0.7, P(B)=0.3, and P(A $\cap$ B)=0.21, are events A and B independent? Why or why not?

2. Jaron has a dozen cupcakes. Three are chocolate with white frosting, three are chocolate with yellow frosting, four are vanilla with white frosting, and two are vanilla with yellow frosting. Are cake flavor and frosting color independent?

3.

	Dance	Sports	TV	Total
Men	2	10	8	20
Women	16	6	8	30
Total	18	16	16	50

The above table represents the favorite leisure activities for 50 adults. Use it to answer the following:

A. Find the probability of your gender.
B. Find the probability of your favorite leisure activity.
C. Find the probability of P(your gender $\cap$ your favorite leisure activity).
D. Are your gender and your favorite leisure activity independent?

VOCABULARY
A probability that takes into account a given condition is called a **conditional probability**. A **given condition** is when we already know the outcome of one of the events. For example, when flipping a coin and rolling a die, the probability of "rolling a 6 given heads", means we already know the coin has resulted in heads. This is written P(6 | heads).

The conditional probability formula is $P\left(A \mid B\right) = \dfrac{P(A \cap B)}{P(B)}$.

Example 4:

A bakery sells vanilla and chocolate cupcakes with white or blue icing.

	White	**Blue**	**Total**
Vanilla	3	5	8
Chocolate	6	7	13
Total	9	12	21

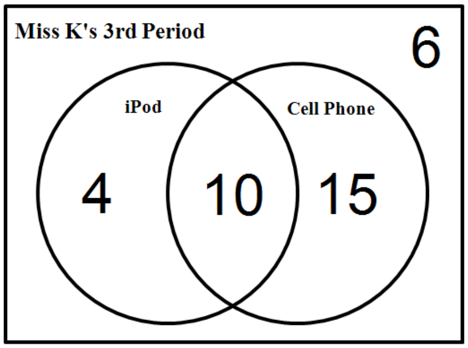
$$P(\text{Vanilla} \mid \text{Blue}) = \frac{P(\text{Vanilla} \cap \text{Blue})}{P(\text{Blue})} = \frac{5}{12}$$

$$P(\text{White} \mid \text{Chocolate}) = \frac{P(\text{White} \cap \text{Chocolate})}{P(\text{Chocolate})} = \frac{6}{13}$$

Alex's favorite cupcake is chocolate with blue icing. What is the probability he will get his favorite cupcake if all the vanilla cupcakes have already been sold?

$$P(\text{Blue} \mid \text{Chocolate}) = \frac{P(\text{Blue} \cap \text{Chocolate})}{P(\text{Chocolate})} = \frac{7}{13}$$

Example 5:



$$P(\text{iPod} \mid \text{Cell Phone}) = \frac{P(\text{iPod} \cap \text{Cell Phone})}{P(\text{Cell Phone})} = \frac{10}{25} = \frac{2}{5}$$

$$P(\text{Cell Phone} \mid \text{No iPod}) = \frac{P(\text{Cell Phone} \cap \text{No iPod})}{P(\text{No iPod})} = \frac{15}{15+6} = \frac{15}{21} = \frac{5}{7}$$

Miss K finds an iPod after class. What is the probability the owner has an iPod and no cell phone?

$$P(\text{iPod} \mid \text{No Cell Phone}) = \frac{P(\text{iPod} \cap \text{No Cell Phone})}{P(\text{No Cell Phone})} = \frac{4}{4+6} = \frac{4}{10} = \frac{2}{5}$$

Practice Exercises E

	Bus	Private Car	Walk	Total
Male	146	166	82	394
Female	154	185	64	403
Total	300	351	146	797

Use the table above to answer the following questions.

1. P(Walk | Female)

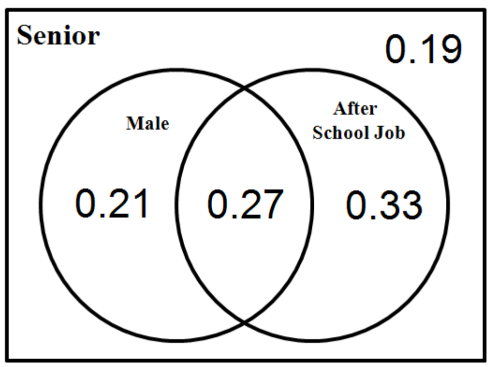
2. P(Male | Private Car)

3. P(Bus | Male)

4. P(Female | Doesn't Walk)

5. What is the probability that Melissa rides the bus? Write the conditional probability equation and then find the probability.

6. Jordan walks to school. What is the probability Jordan is male? Write the conditional probability equation and then find the probability.



Use the Venn diagram above to answer the following questions.

7. P(After School Job | Male)

8. P(Female | No After School Job)

9. P(No After School Job | Male)

10. P(Male | After School Job)

11. Is the probability of having an after school job given you are male the same as the probability of being male given that you have an after school job? Use the probabilities in #7 and #10 to justify your answer.

12. A student works at *McTaco Chimes* what is the probability the student is female?

VOCABULARY
Events A and B are **independent** if and only if they satisfy the probability A given B equals the probability of A OR the probability of B given A equals the probability of B.

$P(A|B) = P(A)$ or $P(B|A) = P(B)$

Example 6:

A bakery sells vanilla and chocolate cupcakes with white or blue icing.

	White	Blue	Total
Vanilla	3	5	8
Chocolate	6	7	13
Total	9	12	21

Are color of icing and cupcake flavor independent?

$$P(Blue|Vanilla) \stackrel{?}{=} P(Blue)$$

$$\frac{P(Blue \cap Vanilla)}{P(Vanilla)} \stackrel{?}{=} P(Blue)$$

$$\frac{5}{8} \stackrel{?}{=} \frac{12}{21}$$

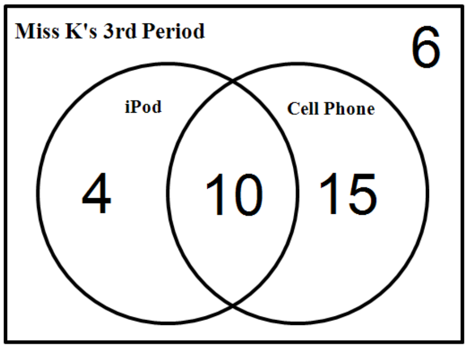
$$0.625 \neq 0.571$$

Therefore the color of icing and cupcake flavor are **not** independent.

Note: Keep in mind the above can also be tested using any of the following options.

1. $P(\text{Blue}|\text{Chocolate}) \stackrel{?}{=} P(\text{Blue})$
2. $P(\text{Vanilla}|\text{Blue}) \stackrel{?}{=} P(\text{Vanilla})$
3. $P(\text{Chocolate}|\text{Blue}) \stackrel{?}{=} P(\text{Chocolate})$
4. $P(\text{White}|\text{Chocolate}) \stackrel{?}{=} P(\text{White})$
5. $P(\text{White}|\text{Vanilla}) \stackrel{?}{=} P(\text{White})$
6. $P(\text{Vanilla}|\text{White}) \stackrel{?}{=} P(\text{Vanilla})$
7. $P(\text{Chocolate}|\text{White}) \stackrel{?}{=} P(\text{Chocolate})$

Example 7:



Are having an iPod and having a cell phone independent?

$$P(\text{iPod}|\text{Cell Phone}) \stackrel{?}{=} P(\text{iPod})$$

$$\frac{P(\text{iPod} \cap \text{Cell Phone})}{P(\text{Cell Phone})} \stackrel{?}{=} P(\text{iPod})$$

$$\frac{10}{25} \stackrel{?}{=} \frac{14}{35}$$

$$\frac{2}{5} = \frac{2}{5}$$

Therefore, having an iPod and having a cell phone are independent.

Practice Exercises F

Students were asked what their main goal for their high school years was. The reported goals were getting good grades, being popular, or excelling at sports.

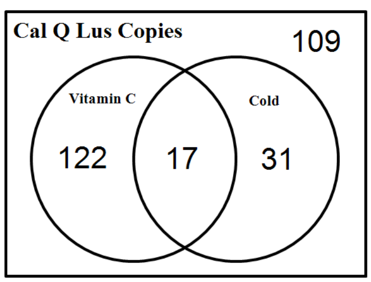
	Goals			
	Grades	**Popular**	**Sports**	**Total**
Boy	117	50	60	227
Girl	130	91	30	251
Total	247	141	90	478

Use the table above to answer the following questions.

1. Is the probability of having good grades as a goal independent of gender?

2. Is gender independent of having popularity as a goal?

3. Workers at Cal Q Lus Copies were polled to see if Vitmain C was a way to reduce the likelihood of getting a cold. According to the diagram, are you less likely to catch a cold if you are taking vitamin C? Justify your answer using conditional probability.



4. Real estate ads suggest that 64% of homes for sale have garages, 21% have swimming pools, and 17% have both features. Are having a garage and having a pool independent events? Justify your answer using conditional probability.

Unit 4 Cluster 2 & 3 Honors (S.CP.8, S.CP.9, S.MD.6, S.MD.7)
Applications of Probability

Cluster 2: Computing probabilities of compound events

- 4.2.3 Apply the general Multiplication Rule
- 4.2.4 Use permutations and combinations to compute probabilities of compound events

Cluster 3: Using probability to evaluate outcomes of decisions

- 4.3.1 Use probability to make fair decisions
- 4.3.2 Analyze decisions and strategies

VOCABULARY

A **compound event** consists of two or more simple events. Tossing a coin is a simple event. Tossing two or more coins is a compound event. A compound event is shown as
$\text{P}(\text{A} \cap B) = P(A) \cdot P(B)$

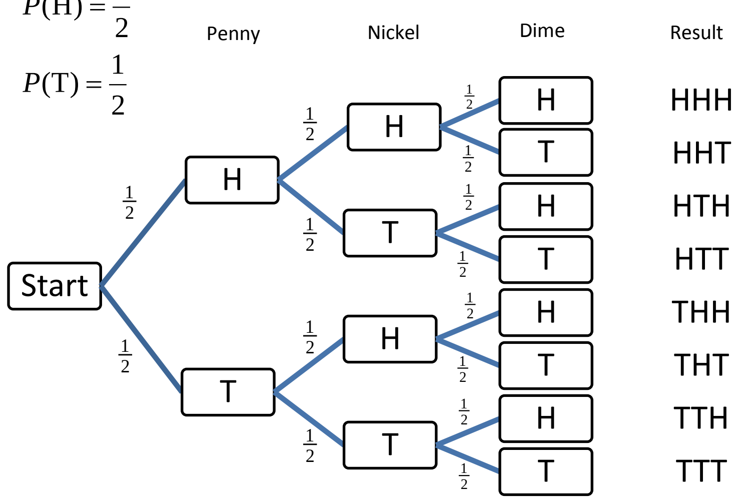
A **tree diagram** is a way of illustrating compound events. Each simple event adds new branches to the tree diagram. The end result shows all possible outcomes.

When events are independent, the probability of a compound event is the product of the probability of the desired outcome for each simple event. This is the general Multiplication Rule.

Example 1:

$P(\text{H}) = \frac{1}{2}$

$P(\text{T}) = \frac{1}{2}$



This is a tree diagram representing the possible outcomes when tossing three different coins.

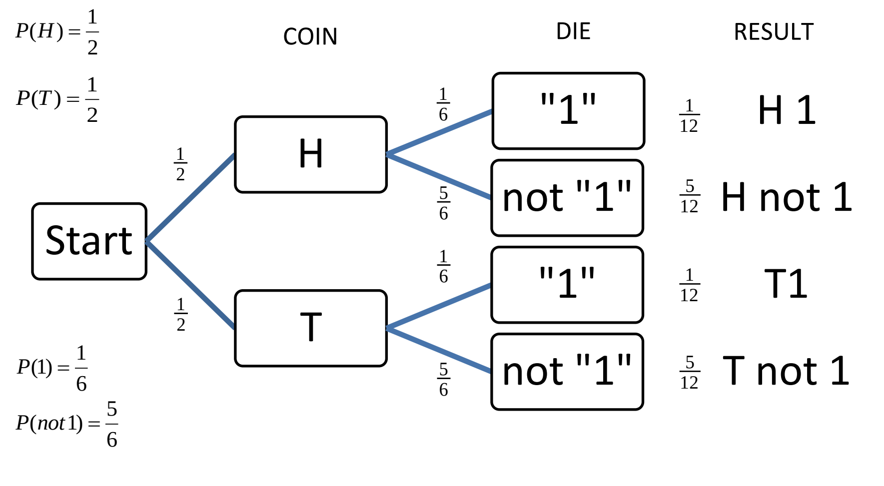
There are eight possible outcomes.

This would be the same representation if looking at the outcomes of tossing one coin three separate times.

$$\text{P(HHH)} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$$\text{P(THH)} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Example 2:



This is a tree diagram representing the possible outcomes when tossing one coin and rolling one die.

$$P(H \cap 1) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

$$P(T \cap not\,1) = \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}$$

Practice Exercises A

Using a tree diagram find the following probabilities.

1. Sophomores are required to either take English 10 or English 10H. They need Secondary Math 2, Secondary Math 2H, or Pre-Calculus. Sophomores also need either Biology or Chemistry.

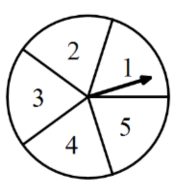
   A. Draw a tree diagram representing all sophomore choices.

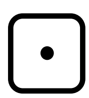
   B. P(Eng10, Sec2H, Chem.)

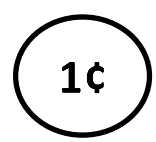
   C. P(Eng10H, Sec2, Bio.)

   D. P(Eng10H, Sec2H, PreCalc)

2. You have the following objects: a spinner with five choices, a six-sided die, and a coin. P(spinner, die, coin)







   A. Draw a tree diagram satisfying the following: a choice on the spinner, a one or two vs. anything else on the die (*Hint*: $P(1 \cup 2)$ vs $P(not\,1 \cup 2)$), and heads or tails.

   B. P(5, 1 or 2, H)

   C. P(even, not 1 or 2, T)

   D. P(1, 1 or 2, H)

   E. P(6, not 1 or 2, T)

   F. P(number < 6, not 1 or 2, H)

   G. P(odd, 1 or 2, T)

VOCABULARY

Probability of **Two Dependent Events:** If two events, *A* and *B*, are dependent, then the probability of both events occurring is $P(A \cap B) = P(A) \cdot P(B|A)$. In other words, the probability of both events occurring is the probability of event A times the probability of event B given that A has already occurred. Likewise, $P(B \cap A) = P(B) \cdot P(A|B)$. These formulas can be extended to any number of independent events.

Example 3:

There are 7 dimes and 9 pennies in a wallet. Suppose two coins are to be selected at random, without replacing the first one. Find the probability of picking a penny and then a dime.

$$\begin{aligned} P(\text{penny, then dime}) &= P(\text{penny}) \cdot P(\text{dime}|\text{first coin was penny}) \\ &= \frac{\text{number of pennies}}{\text{number of coins}} \cdot \frac{\text{number of dimes after a penny has been drawn}}{\text{number of coins after a penny has been drawn}} \\ &= \frac{9}{16} \cdot \frac{7}{15} \\ &= \frac{21}{80} \end{aligned}$$

Example 4:

A basket contains 4 plums, 6 peaches, and 5 oranges. What is the probability of picking 2 oranges, then a peach if 3 pieces of fruit are selected at random?

$$\begin{aligned} P(\text{2 oranges, then peach}) &= P(\text{orange}) \cdot P(\text{orange after orange}) \cdot P(\text{peach after 2 oranges}) \\ &= \frac{5}{15} \cdot \frac{4}{14} \cdot \frac{6}{13} = \frac{120}{2730} = \frac{4}{91} \end{aligned}$$

Example 5:

A cereal company conducts a blind taste test. 55% of those surveyed are women and 45% are men. 75% of the women surveyed like the cereal, and 85% of the men like the cereal. What is the probability that a person selected at random is:

A. A woman who likes the cereal?

B. A man who likes the cereal?

C. A woman who doesn't like the cereal?

D. A man who doesn't like the cereal?

A. $P(\text{woman} \cap \text{like}) = P(\text{woman}) \cdot P(\text{like}|\text{woman})$
$= (0.55)(0.75) = 0.4125$

B. $P(\text{man} \cap \text{like}) = P(\text{man}) \cdot P(\text{like}|\text{man})$
$= (0.45)(0.85) = 0.3825$

C. $P(\text{woman} \cap \text{dislike}) = P(\text{woman}) \cdot P(\text{dislike}|\text{woman})$
$= (0.55)(0.25) = 0.1375$

D. $P(\text{man} \cap \text{dislike}) = P(\text{man}) \cdot P(\text{dislike}|\text{man})$
$= (0.45)(0.15) = 0.0675$

Practice Exercises B

1. A photographer has taken 8 black and white photographs and 10 color photographs for a brochure. If 4 photographs are selected at random, what is the probability of picking first 2 black and white photographs, then 2 color photographs?

2. There are 7 blue pens, 3 black pens, and 2 red pens in a drawer. If you select three pens at random with no replacement, what is the probability that you will select a blue pen, then a black pen, then another blue pen?

3. Tammy's mom is baking cookies for a bake sale. When Tammy comes home, there are 22 chocolate chip cookies, 18 sugar cookies, and 15 oatmeal cookies on the counter. Tammy sneaks into the kitchen, grabs a cookie at random, and eats it. Five minutes later, she does the same thing with another cookie. What is the probability that neither of the cookies was a chocolate chip cookie?

4. There are 2 Root Beers, 2 Sprites, 3 Mountain Dews, and 1 Gatorade left in the vending machine at school. The machines buttons are broken and will randomly give you a random drink no matter what button you push. Find the probability of each outcome?

   A. $P(\text{root beer, root beer})$ B. $P(\text{root beer, mountain dew})$

   C. $P(\text{sprite, gatorade})$ D. $P(\text{mountain dew, mountain dew, mountain dew})$

5. A department store employs high school students, all juniors and seniors. 60% of the employees are juniors. 50% of the seniors are females and 75% of the juniors are males. One student employee is chosen at random. What is the probability of choosing:

   A. A female junior B. A female senior

   C. A male junior D. A male senior

6. There are 400 fans at a baseball game that get popcorn and hotdogs. 75% of the fans getting food are adults and the rest are children. 80% of the children getting food are eating hotdogs and 40% of the adults getting food are getting popcorn. One fan is chosen at random to receive free food. What is the probability of choosing:

   A. An adult with popcorn B. A child with popcorn

   C. An adult with a hotdog D. A child with a hotdog

VOCABULARY

The **factorial function** (symbol: !) is a way to multiply a series of descending natural numbers. $n!=(n)\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$ is the general formula representing a factorial function. For instance, $5!=5\cdot 4\cdot 3\cdot 2\cdot 1=120$. By definition $0!=1$ and $1!=1$.

A **permutation** is a combination, or an arrangement of a group of objects, where order matters. For instance a lock combination or batting orders are examples where the order matters. If we look at the letters A, B, and C there are six ways to arrange the letters, ie: ABC, ACB, BAC, BCA, CBA, CAB.

The number of ways to arrange n distinct objects is indicated by $n!$

For example, in how many orders can a person read 5 different magazines?
This is found by finding 5! or $5\cdot 4\cdot 3\cdot 2\cdot 1=120$. So, there are 120 different orders in which are person can read 5 different magazines.

The number of ways to arrange *n* distinct objects taking them *r* at a time is indicated by $_n\mathrm{P}_r=\dfrac{n!}{(n-r)!}$, where *n* and *r* are whole numbers and $n\geq r$. If $n=r$ then $\mathrm{nPr}=n!$.

For example, in how many orders can a person read 5 magazines selected from a list of 9 possibilities?

This is found by finding $_9\mathrm{P}_5=\dfrac{9!}{(9-5)!}=\dfrac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}=9\cdot 8\cdot 7\cdot 6\cdot 5=15{,}120$. So, there are 15,120 different orders in which a person can ready 5 magazines selected from a list of 9 possibilities.

Permutations with *n* objects where one or more objects repeats, requires taking into consideration each item that is repeated. Use the formula $\dfrac{n!}{s_1!\cdot s_2!\cdots s_k!}$ where s_1 represents the number of times the first object is repeated.

For example, how many way can you arrange the letters in ? KNICKKNACK
There are 10 letters with 4 K's, 2 C's, and 2 N's. The total number of arrangements is
$\dfrac{n!}{s_1!\cdot s_2!\cdots s_k!}=\dfrac{10!}{4!\cdot 2!\cdot 2!}=37{,}800$

Note: All of the probability functions on your graphing calculator can be found by selecting MATH then arrow over to PRB.

Example 6:
You have just purchased 15 new CDs and want to add them to your iPod. You don't want to remove any music already on your iPod and there is only room for 5 more CDs. How many ways can you add 5 different CDs to your iPod?

$_{15}P_5 = \frac{15!}{(15-5)!}$ $= \frac{15!}{10!} = 360,360$ There are 360,360 ways you can add 5 different CDs from your15 choices to your iPod.	$n = 15$ $r = 5$ Use $\text{nPr} = \frac{n!}{(n-r)!}$

Example 7:
Find the number of distinguishable permutations for the word MISSISSIPPI.

$\frac{11!}{4!\cdot 4!\cdot 2!} = 34,650$ There are 34, 650 distinguishable permutations for the word MISSISSIPPI	$n = 11$ $s_1 = \text{I} = 4$ $s_2 = \text{S} = 4$ $s_3 = \text{P} = 2$ Use $\text{nPr} = \frac{n!}{s_1!\cdot s_2!\cdots s_k!}$

Practice Exercises C

Compute.

1. $_8P_3$
2. $_6P_6$
3. $_7P_0$
4. $_{10}P_1$

Find the number of distinguishable permutations for the following words.

5. MATHEMATICS
6. SALT LAKE CITY
7. CHEMISTRY

Solve the following.

8. It is time to elect sophomore class officers. There are 12 people running for four positions: president, vice president, secretary, and historian. How many distinct ways can those positions be filled?

9. You just received 7 new movies in the mail. You only have time to watch 3 this weekend. How many distinct ways can you watch the movies this weekend?

10. The Discriminants are giving a short evening performance. Their latest CD has 14 songs on it; however they only have enough time to perform 8 songs. How many distinct performances can they give?

VOCABULARY

When we are considering **combinations** we are only considering the number of groupings. For instance selecting people to a committee or choosing pizza toppings are examples where the order does not matter.

The number of ways to group *n* distinct objects taking them *r* at a time is indicated by $_{n}C_{r} = \frac{n!}{r!(n-r)!}$, where *n* and *r* are whole numbers and $n \geq r$.

Example 8:

Honors English students are required to read 8 books from a list of 25. How many combinations could a student select?

$_{25}C_{8} = \frac{25!}{8!(25-8)!}$ $= \frac{25!}{8!(17)!}$ $= 1,081,575$ A student has 1,081,575 different groupings of books they could read.	$n = 25$ $r = 8$ Use $_{n}C_{r} = \frac{n!}{r!(n-r)!}$

Practice Exercises D

Compute

1. $_{11}C_{6}$
2. $_{32}C_{0}$
3. $_{65}C_{62}$
4. $_{100}C_{96}$

Solve the following.

5. Four members from a group of 18 on the board of directors at the Fa La La School of Arts will be selected to go to a convention (all expenses paid) in Hawaii. How many different groups of 4 are there?

6. You have just purchased a new video game console. With the purchase you are given the option of obtaining three free games from a selection of ten. How many combinations of games can you choose?

7. You are the manager of a new clothing store. You need 5 new employees and have 20 qualified applicants. How many ways can you staff the store?

You Decide
You are registering for your junior year in school. Your school is on a block schedule, four periods each day. You must take the following courses: English, history, math, and science. You can fill the other four periods with classes of your choice.

English	Math	Science	History	Elective	
English 11	Sec Math 3	Biology	U.S.	Law Enforcement	PE
English 11H	Sec Math 3H	Chemistry	U.S. Honors	World Religions	Sports Medicine
Concurrent	PreCalculus	Physics	AP U.S	European Hist. Craft	Interior Design
AP	Calculus	AP Biology		Psychology	Business
	Concurrent	AP Chem		AP Psychology	Marketing
	Intro. to Stats	AP Physics		Sociology	Web Page Design
	AP Stats	Anatomy		Language	Woods
	Personal Finance	Wild Life Bio		Band	Drafting
		Bio Ag.		Orchestra	Auto
		Genetics		Choir	Sewing
		Astronomy		Dance	Foods
		Geology		Drama	Child Development
		Zoology		Photo	Pre-School
				Ceramics	Financial Lit
				Painting	Green House
				Drawing	Release Time

A. If you are not focusing on the order of your classes, how many different schedules could you construct?

B. Now that you have chosen your classes, how many ways could you set up your schedule?

C. Now that you have chosen your classes, if your Math class must be the first period of the day, how many different schedules can you now have?

Using Permutations and Combinations to Determine Probabilities of Events

Recall
Probability is $P = \dfrac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \dfrac{P(E)}{P(S)}$.

Example 9:

A standard deck of face cards consists of 52 cards, 4 suits (red diamonds, red hearts, black spades, black clubs), and 13 cards in each suit, (numbers 2 through 10, jack, queen, king, ace). What is the probability that the hand consists of 5 red cards?

$P(5 \text{ red cards}) = \dfrac{{}_{26}C_5}{{}_{52}C_5}$ $= \dfrac{65,780}{2,598,960}$ $= 0.025$	The total number of outcomes is ${}_{52}C_5$. There are 26 red cards, so ${}_{26}C_5$ is the number of ways to choose 5 red cards.

Example 10:

Using a standard deck of face cards, what is the probability that the hand consists of 1 diamond?

$P(\text{one diamond}) = \dfrac{{}_{39}C_4 \cdot {}_{13}C_1}{{}_{52}C_5}$ $= \dfrac{82,251 \cdot 13}{2,598,960}$ $= \dfrac{1,069,263}{2,598,960} = 0.411$	The total number of outcomes is ${}_{52}C_5$. There are 39 cards that are not diamonds, so ${}_{39}C_4$ is the number of ways to choose 4 cards that are not diamonds. There are 13 diamonds so ${}_{13}C_1$ is the way to choose 1 diamond.

Practice Exercises E

1. There are 14 black pens and 8 blue pens in a drawer. If 3 pens are chosen at random, what is the probability that they are all blue?

2. Sam has 9 pairs of socks in a drawer: 5 white pairs and 4 gray pairs. If he chooses three pairs at random to pack for a trip, find the probability that he chooses exactly two white pairs.

3. A bag contains 14 cherry, 15 lime, and 10 grape suckers. Find the probability of picking 3 cherry suckers and 2 grape suckers if 5 suckers are chosen at random.

4. Barbara has a collection of 28 movies, including 12 comedies and 16 dramas. She selects 3 movies at random to lend to a friend. What is the probability of her selecting 3 comedies?

5. Five books are chosen at random from a best-seller list that includes 12 novels and 6 biographies. Find the probability of selecting 3 novels, and 2 biographies.

Using Probability Models to Analyze Situations and Make Decisions

VOCABULARY

A game in which all participants have an equal chance of winning is a **fair game**. Similarly, a fair decision is based on choices that have the same likelihood of being chosen.

Fair Decisions and Random Numbers
In order for a decision to be fair each possible outcome must be equally likely. For example if you are hosting a party that includes 20 people and want to randomly choose 5 people to bring treats there are multiple ways in which to make a fair and unfair decision.

A **fair decision** would be to assign each person a number. Write each of these numbers on a separate piece of paper and drop them into a hat. Shuffle the numbers in the hat and choose five papers to match five party members. These five people would be assigned to bring treats. This is often referred to as "drawing lots".

An **unfair decision** would be to write each person's name on a piece of paper. Then arrange the papers in alphabetical order. Starting with the first paper, flip a coin and record the result: heads or tails. The first five friends that have a tail flipped for them must bring treats. This is unfair because you will probably never flip a coin for your friends that have names that begin with letters at the end of the alphabet.

Often **Random Numbers** are used to help you make fair decisions. For an event to be random there is no pre-determined bias towards any particular outcome. Often people use random number tables, random number generators on a calculator, or simply shuffle pieces of papers that have numbers printed on them.

<u>Example 11:</u>
The addition game is played by rolling two dice. Player 1 gets a point if the sum of the two dice is even. Player 2 gets a point if the sum of the two dice is odd. Use probability to determine if this game is fair.

<table>
<tr><td>There are six possible outcomes for each dice.

Using the general multiplication rule $6 \cdot 6 = 36$.</td><td>Start by determining how many different outcomes can occur when rolling two dice.</td></tr>
<tr><td>Rolling Sums that are Even
<table>
<tr><td>1, 1</td><td>2, 2</td><td>3, 1</td><td>4, 2</td><td>5, 1</td><td>6, 2</td></tr>
<tr><td>1, 3</td><td>2, 4</td><td>3, 3</td><td>4, 4</td><td>5, 3</td><td>6, 4</td></tr>
<tr><td>1, 5</td><td>2, 6</td><td>3, 5</td><td>4, 6</td><td>5, 5</td><td>6, 6</td></tr>
</table></td><td>Determine the number of ways that you can roll an even number and an odd number by listing all possible outcomes</td></tr>
</table>

Rolling Sums that are Odd

1, 2	2, 1	3, 2	4, 1	5, 2	6, 1
1, 4	2, 3	3, 4	4, 3	5, 4	6, 3
1, 6	2, 5	3, 6	4, 5	5, 6	6, 5

There are 18 ways to roll an even sum and 18 ways to roll an odd sum.

$P(\text{even}) = \frac{18}{36} = 0.50$ $P(\text{odd}) = \frac{18}{36} = 0.50$	Calculate the probability of rolling an even sum. Calculate the probability of rolling an odd sum.

This game is fair because each player has the same probability of rolling and even or odd number.

<u>Example 12:</u>

The multiplication game is played by rolling two dice. Player 1 gets a point if the product of the two dice is even and player 2 gets a point if the product of the two dice is odd. Use probability to determine if this game is fair.

There are six possible outcomes for each dice. Using the general multiplication rule $6 \cdot 6 = 36$.	Start by determining how many different outcomes can occur when rolling two dice.

Determine the number of ways that you can roll and even number and an odd number by listing all possible outcomes

Rolling a Product that is Even

1, 2	2, 1	3, 2	4, 1	5, 2	6, 1
1, 4	2, 2	3, 4	4, 2	5, 4	6, 2
1, 6	2, 3	3, 6	4, 3	5, 6	6, 3
	2, 4		4, 4		6, 4
	2, 5		4, 5		6, 5
	2, 6		4, 6		6, 6

Rolling a Product that is Even

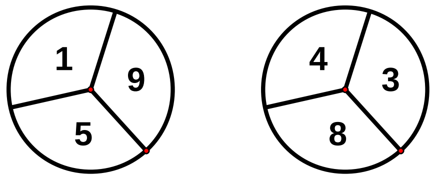
1, 1	3, 1	5, 1
1, 3	3, 3	5, 3
1, 5	3, 5	5, 5

There are 27 ways to roll a product that is even and 9 ways to roll a product that is odd.

$P(\text{even}) = \frac{27}{36} = 0.75$ $P(\text{odd}) = \frac{9}{36} = 0.25$	Calculate the probability of rolling an even product. Calculate the probability of rolling an odd product.

This game is not fair because the probability of rolling an even product is higher than rolling an odd product.

Example 13:
Suppose each player spins the spinner once. Player A using spinner A, and Player B using spinner B. The one with the larger number wins.



Spinner A　　Spinner B

To determine how many possible outcomes there are, list all the outcomes or use the general multiplication rule. 1, 4　　1, 3　　1, 8 5, 4　　5, 3　　5, 8 9, 4　　9, 3　　9, 8 $3 \cdot 3 = 9$ There are 9 possible outcomes.	Start by determining how many possible outcomes there are.
List the number of ways in which player A wins 5, 4　　5, 3　　9, 4　　9, 3　　9, 8 List the number of ways in which play B wins 1, 4　　1, 3　　1, 8　　5, 8	Count the number of ways in which player A wins and the number of ways in which player B wins.
$P(A) = \frac{\text{number of ways player A wins}}{\text{total number of outcomes}} = \frac{5}{9} = .\overline{5}$ $P(B) = \frac{\text{number of ways player B wins}}{\text{total number of outcomes}} = \frac{4}{9} = .\overline{4}$	Find the probability that player A will win or *P(A)* and the probability that player B will win or *P(B)*.
This game is not fair because player A has a higher probability of winning.	

Using Probability to Analyze Decisions

Understanding the probability or expected outcomes from probability models and experiments can help us make good decisions.

Example 14:

Mr. Green created a frequency table to collect data about his students and how their study habits related to performance in his class. The table is shown below.

	Studied	Did not Study	Totals
Passed	16	6	22
Failed	2	12	14
Totals	18	18	36

What is the probability that a student who studies will pass the class?

What is the probability that a student that does not study will pass his class?

Based off of these probabilities, if you want to pass the class should you study or not?

Find $P(\text{studied} \mid \text{passed})$ and $P(\text{did not study} \mid \text{passed})$.

$$P(\text{studied} \mid \text{passed}) = \frac{\text{number of students that studied and passed}}{\text{total number of students that studied}} = \frac{16}{18} = 0.\bar{8}$$

$$P(\text{did not study} \mid \text{passed}) = \frac{\text{number of students that did not study and passed}}{\text{total number of students that studied}} = \frac{6}{18} = .\bar{3}$$

There is a higher probability that a student will pass the class if they study

Example 15:

A teacher is conducting an action research project to determine the effectiveness of an instructional strategy. The new instructional strategy was used with two class periods of 45 students each, and the traditional teaching method was used with 2 class periods of 45 students each. Students were given a pre- and a post-exam to determine whether or not they improved after the instruction. The results are shown in the table below.

	Improved	Did not Improve	Total
Received the new strategy	78	12	90
Received the traditional method	35	55	90
Totals	123	67	180

What is the probability that a student improved given that he was instructed with the new strategy?

What is the probability that a student received the traditional method of instruction given that he did not improve?

If the teacher decided that the new instructional strategy was more effective than the traditional method of teaching, did she make a good decision?

Find $P(\text{improved}\,|\,\text{new})$ and $P(\text{traditional}\,|\,\text{no improvement})$.

$$P(\text{improved}\,|\,\text{new}) = \frac{\text{number of students that improved with the new stratgy}}{\text{total number of students with new strategy}} = \frac{78}{90} = 0.8\overline{6}$$

$$P(\text{traditional}\,|\,\text{no improvement}) = \frac{\text{number of students that did not improve with traditional}}{\text{total number of students that did not improve}}$$

$$P(\text{traditional}\,|\,\text{no improvement}) = \frac{55}{67} = .821$$

There is a higher probability that a student will improve from the pre- to the post-exam if the new instructional strategy is used.

Practice Exercises F

The table below shows the number of students at a certain high school who took an ACT preparatory class before taking the ACT exam and the number of students whose scores were at or above the minimum requirement for college entrance.

	Prep Class	No Prep Class	Totals
At or above minimum requirement	268	210	478
Below minimum requirement	57	115	172
Totals	325	325	650

1. What is the probability that a student scored at or above the minimum requirement for college entrance given that he or she took the ACT preparatory class?

2. What is the probability that a student scored below the minimum requirement given that he or she did not take the ACT preparatory class?

3. A student decides not to take the preparatory class before taking the ACT exam. Is this a good decision? Explain your answer.

Unit 5

Similarity, Right Triangle Trigonometry, and Proof

Unit 5 Cluster 1 (G.SRT.1, G.SRT.2, G.SRT.3)
Understand Similarity in terms of similarity transformations

Cluster 1: Understanding similarity in terms of transformations

- 5.1.1: Dilation with a center and scale factor, with a parallel line and a line segment
- 5.1.2: Transformations with similarity using equality of corresponding angles and proportionality of corresponding sides
- 5.1.3: Establish criterion of AA using similarity transformations

VOCABULARY

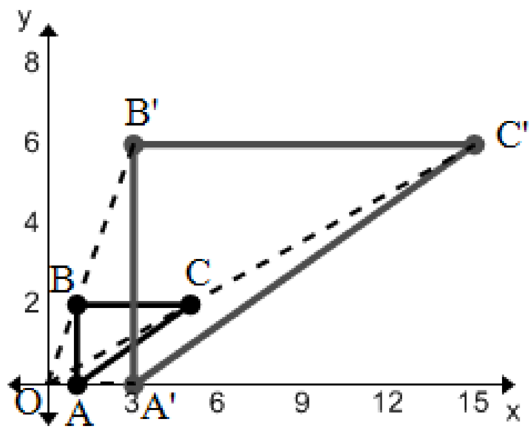
A **dilation** is a transformation that produces an image that is the same shape as the original figure but the image is a different size. The dilation uses a center and a scale factor to create a proportional figure.

The **center of dilation** is a fixed point in the plane about which all points are expanded or contracted.

The **scale factor** is the ratio of the new image to the original image (i.e. if the original figure has a length of 2 and the new figure has a length of 4, the scale factor is $\frac{4}{2}=2$.)

$\Delta A'B'C'$ (the image) is a dilation of ΔABC (the pre-image) with a scale factor of 3.

The origin, point *O*, is the center of dilation.



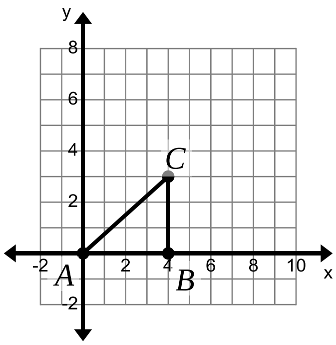
$A(1,0)$	$A'(3,0)$
$B(1,2)$	$B'(3,6)$
$C(5,2)$	$C'(15,6)$

The ratio of the lengths from the center of dilation to each coordinate is equal to the scale factor.

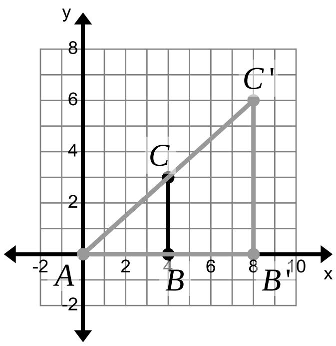
$$\frac{OA'}{OA}=\frac{OB'}{OB}=\frac{OC'}{OC}=3$$

Example 1: Center at the origin

Dilate the triangle with vertices $A(0,0)$, $B(4,0)$ and $C(4,3)$ by a scale factor of 2 and center at $(0,0)$.



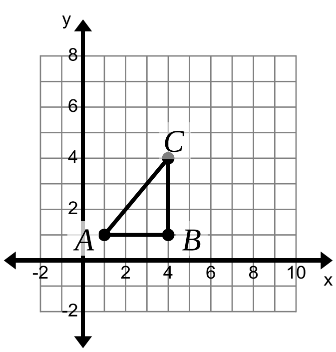
Draw the triangle and label its vertices.



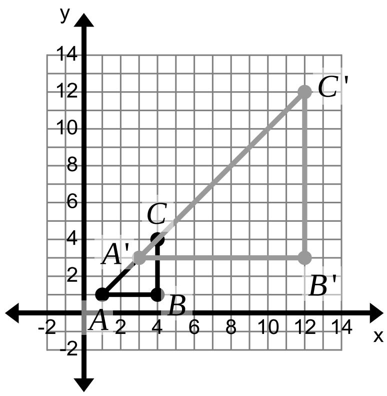
$\overline{A'B'}$ will be twice as long as $\overline{AB}$. Since the center of the dilation is at $(0,0)$, A' is mapped to $(0,0)$ while B' is mapped to $(8,0)$. This makes $\overline{A'B'} = 8$ which is twice the length of $\overline{AB}$. Similarly, $\overline{A'C'}$ will be twice as long as $\overline{AC}$, mapping C' to $(8,6)$. $\overline{B'C'}$ will be twice as long as $\overline{BC}$.

Example 2: Center at the origin

Dilate the triangle with vertices $A(1,1)$, $B(4,1)$ and $C(4,4)$ by a scale factor of 3 and center at $(0,0)$.



Draw the triangle and label its vertices.

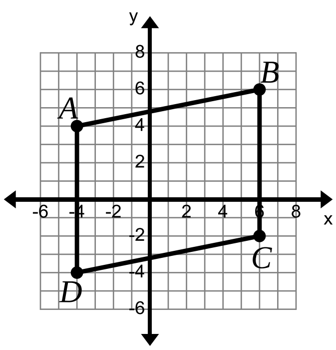


$\overline{A'B'}$ will be three times as long as $\overline{AB}$. Since the center of the dilation is at $(0,0)$, A' is mapped to $(3,3)$ while B' is mapped to $(12,3)$. This makes $\overline{A'B'}=9$ which is three times the length of $\overline{AB}$. Similarly, $\overline{A'C'}$ will be three times as long as $\overline{AC}$, mapping C' to $(12,12)$. $\overline{B'C'}$ will be three times as long as $\overline{BC}$.

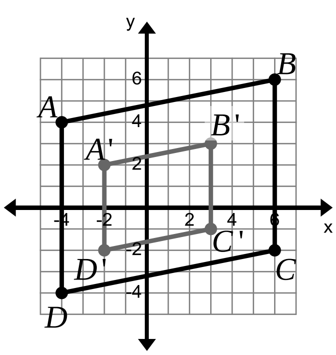
Dilations can be performed on other shapes besides triangles. The shapes can be in any of the four quadrants or even in more than one quadrant. The dilation can also shrink the original shape instead of enlarging it.

<u>Example 3: Center at the origin</u>

Dilate a parallelogram with vertices $A(-4,4)$, $B(6,6)$, $C(6,-2)$ and $D(-4,-4)$ by a scale factor of $\dfrac{1}{2}$ and center at $(0,0)$.



Draw the parallelogram and label its vertices.



Coordinates of Vertices

$A(-4,4)\rightarrow A'(-2,2)$

$B(6,6)\rightarrow B'(3,3)$

$C(6,-2)\rightarrow C'(3,-1)$

$D(-4,-4)\rightarrow D'(-2,-2)$

Measure of Side Lengths

$\left|\overline{AB}\right|=2\sqrt{26},\ \left|\overline{A'B'}\right|=\sqrt{26}$

$\left|\overline{CD}\right|=2\sqrt{26},\ \left|\overline{C'D'}\right|=\sqrt{26}$

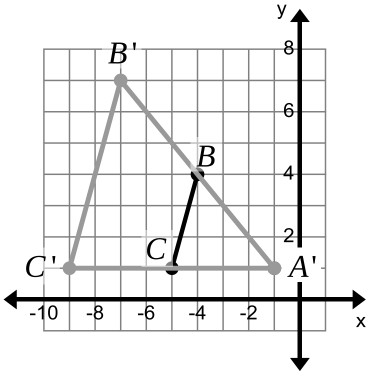
$\left|\overline{AD}\right|=8,\ \left|\overline{A'D'}\right|=4$

$\left|\overline{BC}\right|=8,\ \left|\overline{B'C'}\right|=4$

If the center of the dilation is not at the origin, then you will want to use graph paper and rulers or dynamic geometry software such as: Geogebra or Geometer's Sketchpad. Geogebra is a free download and can be found at http://www.geogebra.org/cms/.

Example 4: Center not at the origin

Dilate the triangle with vertices $A(-1,1)$, $B(-4,4)$ and $C(-5,1)$ by a scale factor of 2 and center at $(-1,1)$.



Coordinates of Vertices

$A(-1,1) \rightarrow A'(-1,1)$

$B(-4,4) \rightarrow B'(-7,7)$

$C(-5,1) \rightarrow C'(-9,1)$

Measure of Side Lengths

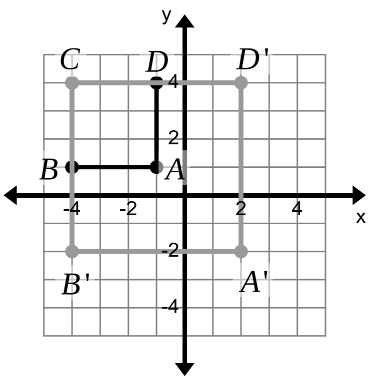
$\left|\overline{AB}\right| = 3\sqrt{2}$, $\left|\overline{A'B'}\right| = 6\sqrt{2}$

$\left|\overline{AC}\right| = 4$, $\left|\overline{A'C'}\right| = 8$

$\left|\overline{CB}\right| = \sqrt{10}$, $\left|\overline{C'B'}\right| = 2\sqrt{10}$

Example 5: Center not at the origin

Dilate the square with vertices $A(-1,1)$, $B(-4,1)$, $C(-4,4)$ and $D(-1,4)$ by a scale factor of 2 and center at $(-4,4)$.



Coordinates of Vertices

$A(-1,1) \rightarrow A'(2,-2)$

$B(-4,1) \rightarrow B'(-4,-2)$

$C(-4,4) \rightarrow C'(-4,4)$

$C(-1,4) \rightarrow C'(2,4)$

Measure of Side Lengths

$\left|\overline{AB}\right| = 3$, $\left|\overline{A'B'}\right| = 6$

$\left|\overline{AD}\right| = 3$ $\left|\overline{A'D'}\right| = 6$

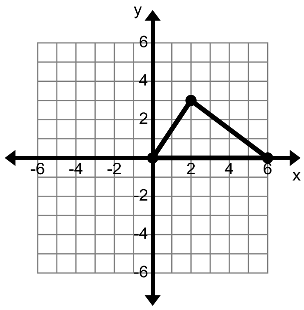
$\left|\overline{BC}\right| = 3$, $\left|\overline{B'C'}\right| = 6$

$\left|\overline{CD}\right| = 3$, $\left|\overline{C'D'}\right| = 6$

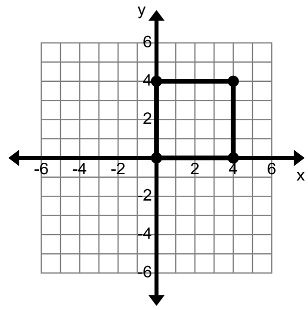
Practice Exercises A

Draw the dilation image of each figure with given center and scale factor.

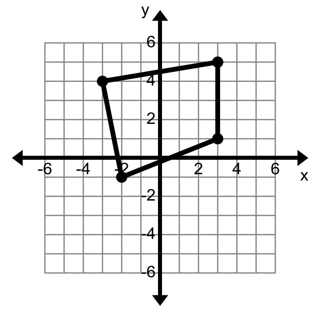
1. Center $(0,0)$; scale factor 3



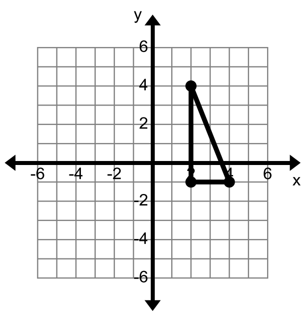
2. Center $(0,0)$; scale factor 2



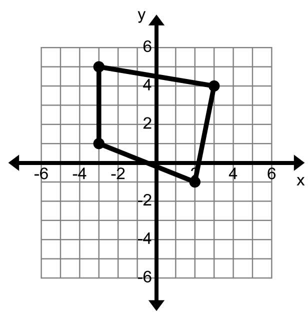
3. Center $(0,0)$; scale factor $\frac{1}{2}$



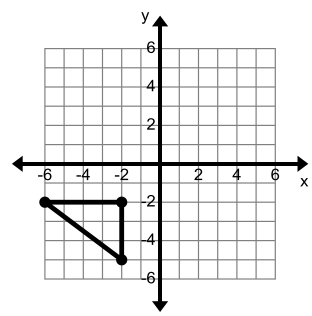
4. Center $(4,-1)$; scale factor 2



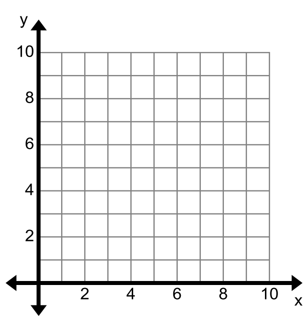
5. Center $(3,4)$; scale factor 3



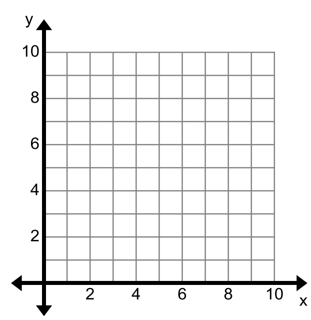
6. Center $(-2,-5)$; scale factor 2



7. Graph the pre-image with given vertices. $J(2,4)$, $K(4,4)$, and $P(3,2)$. Then graph the image with center of dilation at the origin and a scale factor of 2.



8. Graph the pre-image with given vertices. $J(2,4)$, $K(4,4)$, and $P(3,2)$. Then graph the image with center of dilation at the origin and a scale factor of $\frac{1}{2}$.



Determine whether each statement is true or false.

9. A dilation with a scale factor greater than 1 will shrink the image.

10. For a dilation, corresponding angles of the image and pre-image are congruent.

11. A dilation image cannot have any points in common with its pre-image.

12. A dilation preserves length.

Similarity

VOCABULARY

Two figures are **similar** if and only if there is a **dilation** that maps one figure onto the other. In the new figure, corresponding angles are congruent and corresponding sides are proportional to the original figure. You can denote that two figures are similar by using the symbol $\sim$. For example, $\triangle ABC \sim \triangle DEF$.

Optional Exploration Activity

Is it sufficient to know that two angles are congruent to two corresponding angles in another triangle in order to conclude that the two triangles are similar?

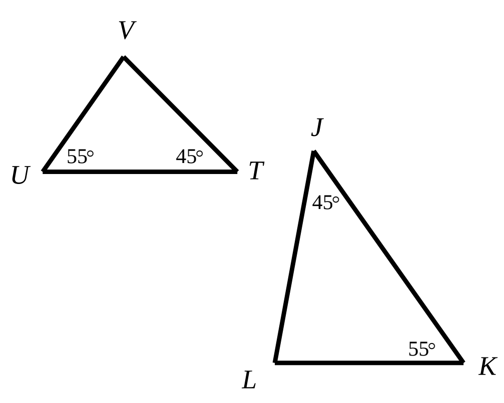
Step 1: Using graph paper, dynamic geometry software or patty paper, have students construct any triangle and label its vertices.

Step 2: Construct a second triangle with two angles that measure the same as two angles in the first triangle.

Step 3: Measure the lengths of the sides of both triangles and compare the ratios of the corresponding sides.

Step 4: Compare your results to those of other students.

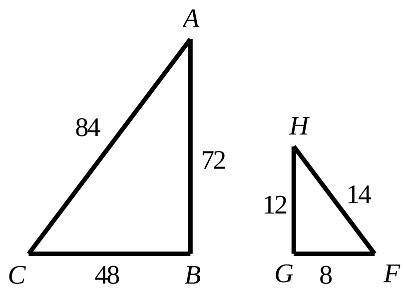
Example 1: State if the triangles in each pair are similar. If so, state how you know they are similar and create the similarity statement.



Since $\angle VUT \cong \angle LKJ$, $\angle VTU \cong \angle LJK$, and the angles of a triangle add to $180°$, the third set of angles must also be congruent. The angles are congruent which forces the corresponding sides to be proportional.

$\triangle TUV \sim \triangle JKL$ because the corresponding angles are congruent.

<u>Example 2:</u> State if the triangles in each pair are similar. If so, state how you know they are similar and create the similarity statement.



Find the sets of corresponding sides and show that they have the same ratio of proportionality.

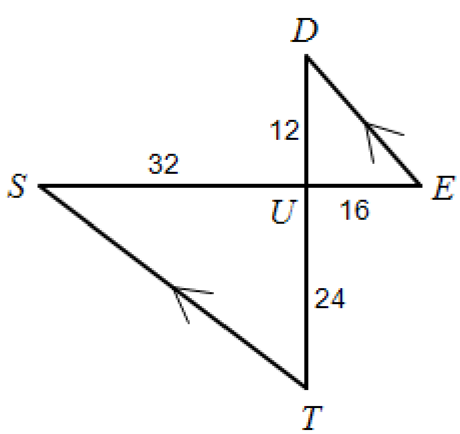
$\overline{AC}$ looks similar to $\overline{HF}$, $\frac{\overline{AC}}{\overline{HF}} = \frac{84}{14} = 6$

$\overline{BC}$ looks similar to $\overline{GF}$, $\frac{\overline{BC}}{\overline{GF}} = \frac{48}{8} = 6$

$\overline{AB}$ looks similar to $\overline{HG}$, $\frac{\overline{AB}}{\overline{HG}} = \frac{72}{12} = 6$

$\triangle ABC \sim \triangle HGF$ because the sets of corresponding sides are proportional which will force the angles to be congruent. This is SSS similarity.

<u>Example 3:</u> State if the triangles in each pair are similar. If so, state how you know they are similar and create the similarity statement.



$\overline{DE} \parallel \overline{ST}$, therefore $\angle D \cong \angle T$ because alternate interior angles are congruent.

$\angle EUD \cong \angle SUT$ because they are vertical angles.

By AA similarity $\angle SUT \sim \angle EUD$.

You can also show that the corresponding sides are proportional.

$\overline{SU}$ looks similar to $\overline{EU}$, $\frac{\overline{SU}}{\overline{EU}} = \frac{32}{16} = 2$

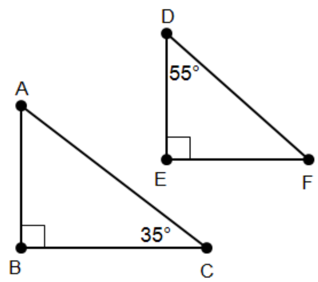
$\overline{TU}$ looks similar to $\overline{DU}$, $\frac{\overline{TU}}{\overline{EU}} = \frac{24}{12} = 2$

$\triangle SUT \sim \triangle EUD$ because the sets of corresponding sides are proportional and the included angle is congruent. This is called SAS similarity.

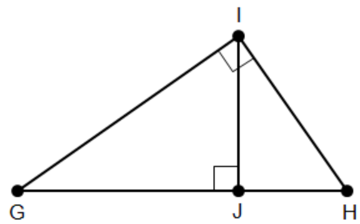
Practice Exercises B

Are the following triangles similar? If so, write a similarity statement.

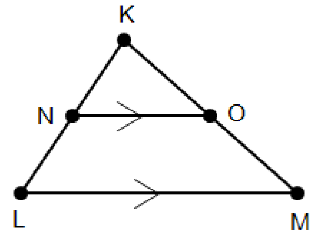
1.



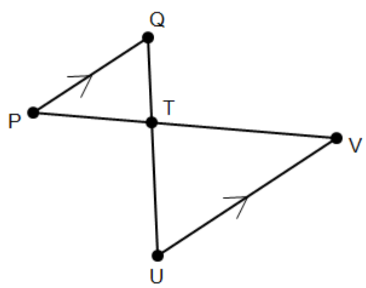
2.



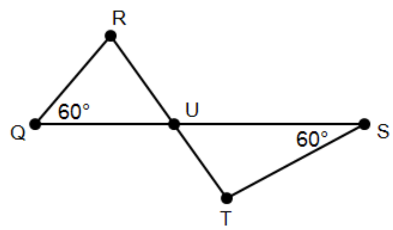
3.



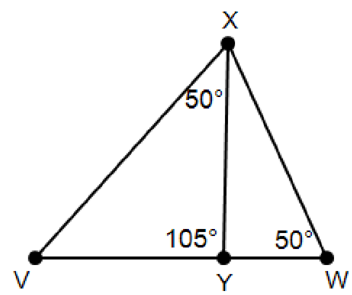
4.



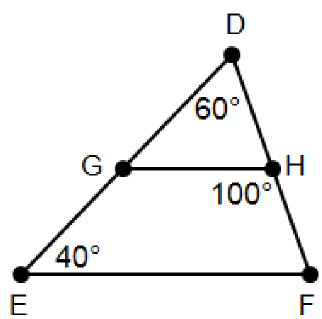
5.



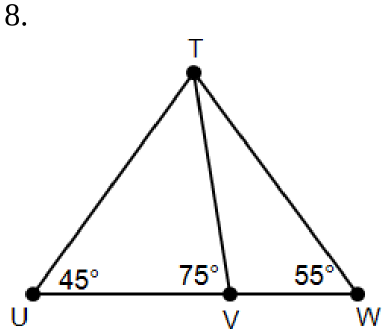
6.



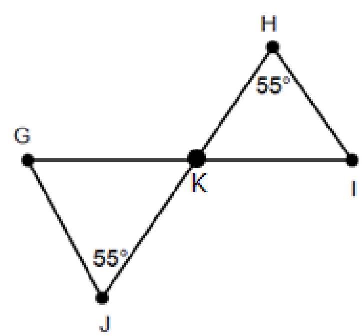
7.



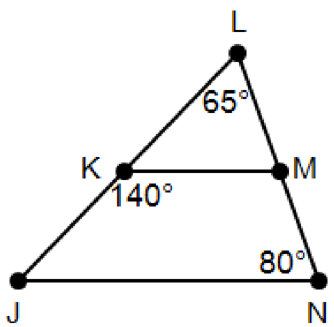
8.



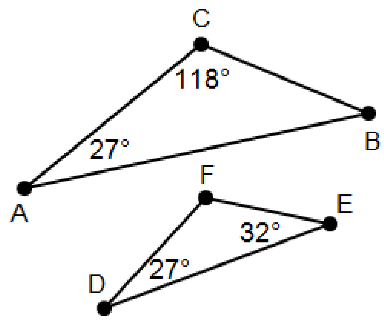
9.



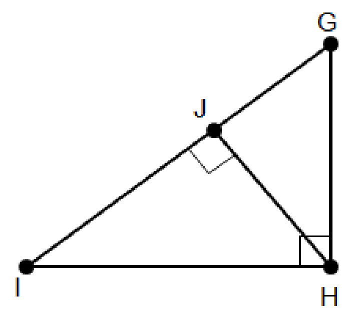
10.



11.



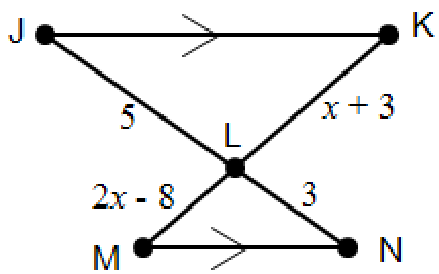
12.



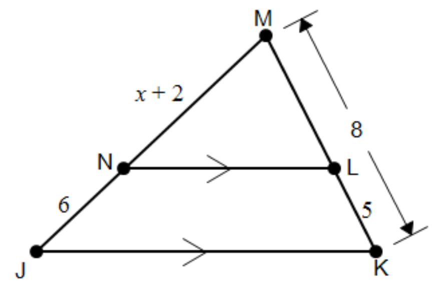
Practice Exercises C

Identify the similar triangles and write a similarity statement. Then find the value for *x* and the measure of the indicated sides.

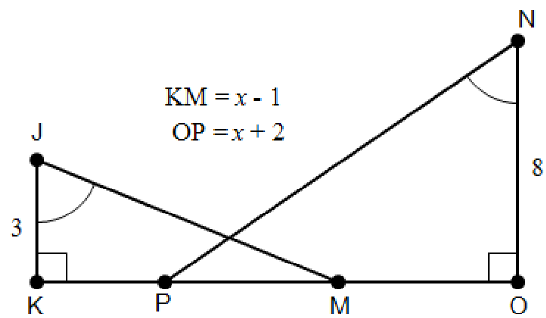
1. ML and LK



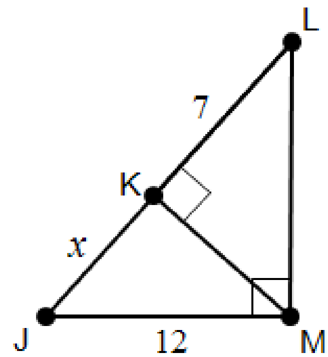
2. MN and MJ



3. KM and OP



4. JK and JL



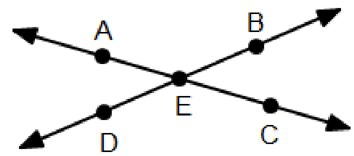
Unit 5 Cluster 2 (G.CO.9)
Prove Theorems about Lines and Angles

Cluster 2: Prove Geometric Theorems

5.2.1 Prove theorems about lines and angles: vertical angles, alternate interior angles, corresponding angles, points on a perpendicular bisector of a segment

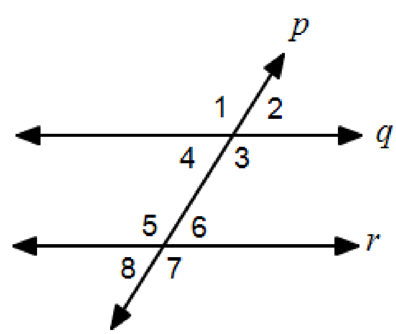
VOCABULARY

Two nonadjacent angles formed by two intersecting lines are called **vertical angles** or **opposite angles**. For example, $\angle AEB$ and $\angle DEC$ are vertical angles.



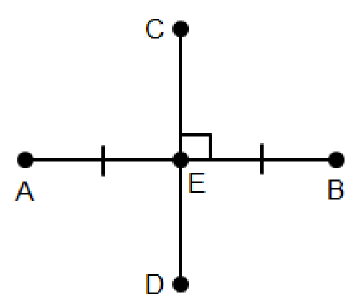
A line that intersects two or more lines in a plane at different points is called a **transversal**. Transversal *p* intersects lines *q* and *r*.

When two parallel lines are intersected by a transversal, angles that are in the same position at each intersection are called **corresponding angles**. For example, $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$ are all corresponding angles.



When two parallel lines are intersected by a transversal, angles that are inside of the two parallel lines, but on opposite sides of the transversal are called **alternate interior angles**. For example, $\angle 3$ and $\angle 5$, and $\angle 4$ and $\angle 6$ are both alternate interior angles.

A segment, line, or ray perpendicular to a given segment that cuts the segment into two congruent parts is called a **perpendicular bisector**. For example, $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$.



Theorems and Postulates

This unit of the core requires students to create proofs, formal or informal, to prove the following theorems and postulates. Example proofs will be provided; however, teachers will probably do tasks with students to complete these proofs in class.

Vertical Angle Theorem: If two angles are vertical angles, then they are congruent.

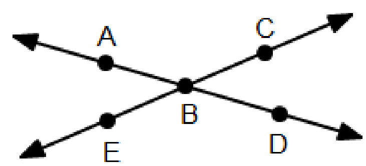
Corresponding Angle Postulate: If two parallel lines are cut by a transversal, then each pair of corresponding angles is congruent.

Alternate Interior Angle Theorem: If two parallel lines are cut by a transversal, then each pair of alternate interior angles is congruent.

Perpendicular Bisector Theorem: Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

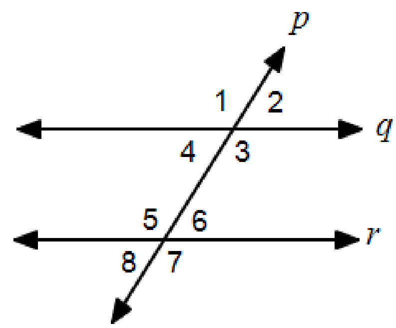
<u>Example 1:</u> Proof of the Vertical Angle Theorem

Given: $\angle ABC$ and $\angle EBD$ are vertical angles.
Prove: $\angle ABC \cong \angle EBD$



$\angle ABC$ and $\angle CBD$ are vertical angles.	Given.
$\angle ABC$ is supplementary to $\angle CBD$ and $\angle EBD$ is supplementary to $\angle CBD$.	$\angle ABC$ and $\angle CBD$ form a straight angle. $\angle EBD$ and $\angle CBD$ form a straight angle.
$m\angle ABC + m\angle CBD = 180°$ $m\angle EBD + m\angle CBD = 180°$	Definition of supplementary angles.
$m\angle ABC = 180° - m\angle CBD$ $m\angle EBD = 180° - m\angle CBD$	Subtraction property of equality.
$m\angle ABC = m\angle EBD$	Transitive property.

Example 2: Proof of Corresponding Angle Postulate



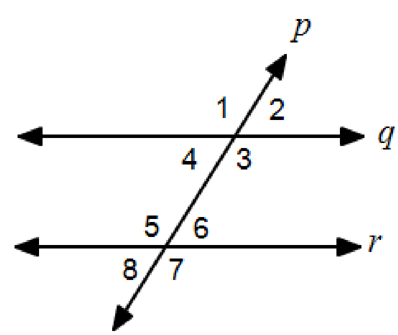
Given: $q \parallel r$, $\angle 1$ and $\angle 5$ are corresponding angles
Prove: $\angle 1 \cong \angle 5$

We know that angle $\angle 4$ is supplementary to angle $\angle 1$ from the straight angle theorem. $\angle 5$ and $\angle 4$ are also supplementary, because they are interior angles on the same side of transversal p (same side interior angles theorem).

Therefore, since $m\angle 4 = 180° - m\angle 1 = 180° - m\angle 5$, we know that $m\angle 1 = m\angle 5$ which means that $\angle 1 \cong \angle 5$. This can be proven for every pair of corresponding angles in the same way.

Example 3: Proof of Alternate Interior Angle Theorem

Given: $q \parallel r$
Prove: $\angle 3 \cong \angle 7$



$\angle 1 \cong \angle 3$ and $\angle 5 \cong \angle 7$ because they are vertical angles.

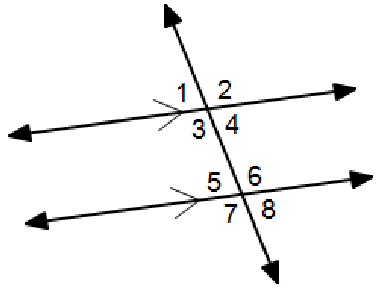
If $q \parallel r$ then $\angle 1 \cong \angle 5$ and $\angle 3 \cong \angle 7$ because they are corresponding angles.

Therefore, using the transitive property, $\angle 3 \cong \angle 7$.

A similar argument can be given to show $\angle 4 \cong \angle 6$.

Practice Exercises A

Use the figure below for problems 1–2.



1. Identify the pairs of angles that are vertical angles, corresponding angles, and alternate interior angles.

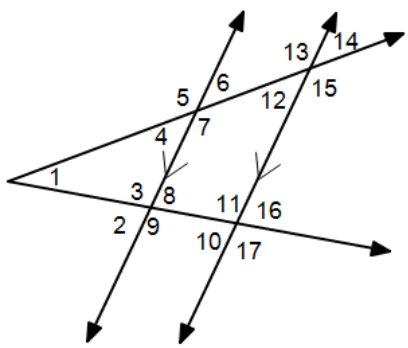
Vertical Angles	Corresponding Angles	Alternate Interior Angles

2. Given $m\angle 1 = 72°$, find the measure of the remaining angles.

$m\angle 2 =$ $m\angle 3 =$ $m\angle 4 =$ $m\angle 5 =$

$m\angle 6 =$ $m\angle 7 =$ $m\angle 8 =$

Use the figure below for problems 3–4.



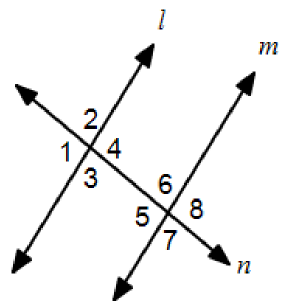
3. Identify the pairs of angles that are vertical angles, corresponding angles, and alternate interior angles.

Vertical Angles	Corresponding Angles	Alternate Interior Angles

4. Given $m\angle 5 = 110°$ and $m\angle 17 = 95°$, find the measure of the remaining angles.

$m\angle 1 =$	$m\angle 2 =$	$m\angle 3 =$	$m\angle 4 =$	$m\angle 6 =$
$m\angle 7 =$	$m\angle 8 =$	$m\angle 9 =$	$m\angle 10 =$	$m\angle 11 =$
$m\angle 12 =$	$m\angle 13 =$	$m\angle 14 =$	$m\angle 15 =$	$m\angle 16 =$

Use the figure at the right for questions 5–7.

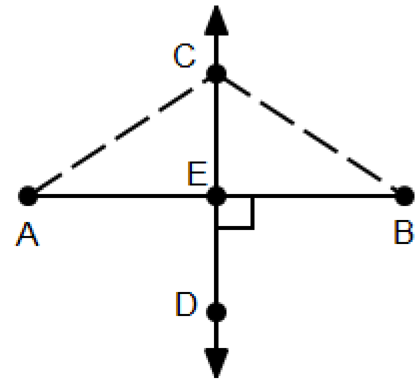


5. Given: $l \parallel m$ prove that $\angle 2 \cong \angle 7$.

6. Given: $l \parallel m$ prove that $m\angle 3 + m\angle 5 = 180°$.

7. Given: $l \parallel m$ prove that $m\angle 2 + m\angle 8 = 180°$.

<u>Example:</u> Proof of the Perpendicular Bisector Theorem

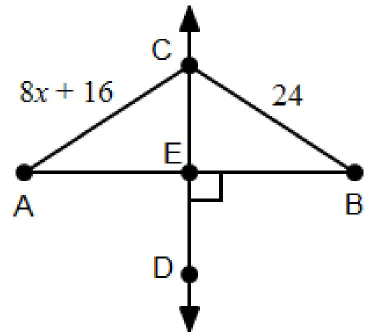


Given: $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$
Prove: $\overline{AC} \cong \overline{BC}$

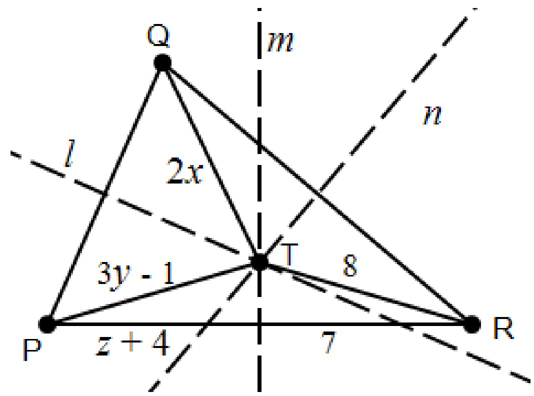
Because $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$, $\overline{AE} \cong \overline{BE}$. $\angle CEA \cong \angle CEB$ because they are both right angles. $\overline{CE}$ is congruent to itself because of the reflexive property. This means that $\triangle CEA \cong \triangle CEB$ using SAS. Since the triangles are congruent, all of their corresponding parts must be congruent. Therefore, $\overline{AC} \cong \overline{BC}$.

Practice Exercises B

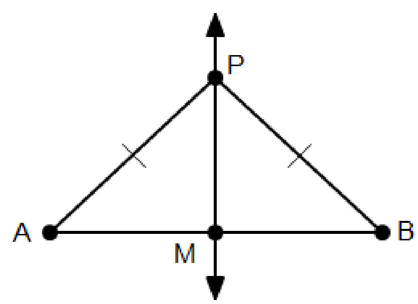
1. $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$. Solve for *x*.



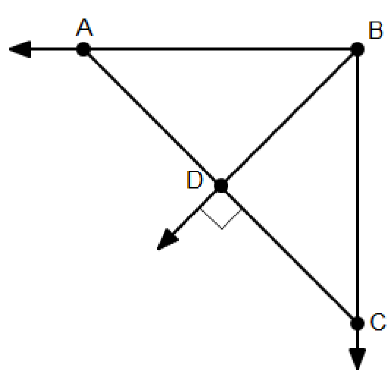
2. Lines *l*, *m*, and *n* are perpendicular bisectors of ΔPQR and meet at T. If $TQ = 2x$, $PT = 3y - 1$, and $TR = 8$, find *x*, *y*, and *z*.



3. Given that $PA = PB$ and $\overline{PM} \perp \overline{AB}$ at *M*, prove that $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$.



4. Given that $\overrightarrow{BD}$ is the perpendicular bisector of $\overleftrightarrow{AC}$, prove that $\triangle ABD \cong \triangle CBD$.



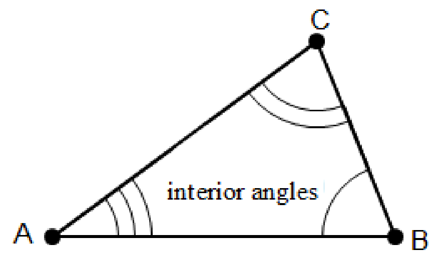
Unit 5 Cluster 2 (G.CO.10)
Prove Theorems about Triangles

Cluster 2: Prove Geometric Theorems

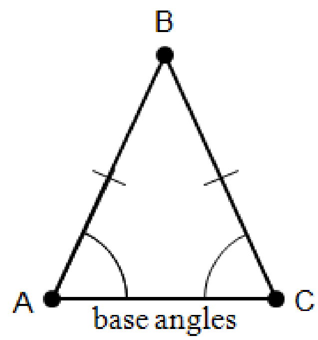
5.2.2 Prove theorems about triangles: sum of interior angles, base angles of isosceles triangles, segment joining midpoints of two sides is parallel to third side and half the length, and medians meet at a point

VOCABULARY

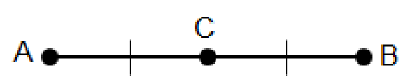
The angles inside of a triangle are called **interior angles**.



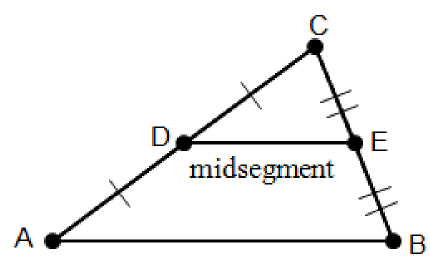
A triangle with at least two congruent sides is called an **isosceles triangle**. In an isosceles triangle, the angles that are opposite the congruent sides are called **base angles**.



A point that is halfway between the endpoints of a segment is called the **midpoint**. Point C is the midpoint of $\overline{AB}$.

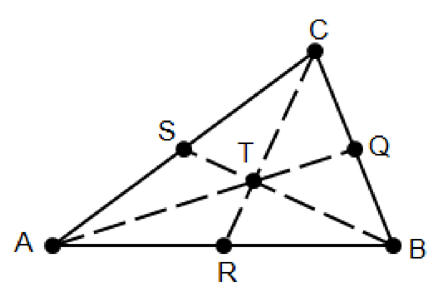


A segment whose endpoints are the midpoints of two sides of a triangle is called the **midsegement of a tiangle**.



The line connecting midpoints to the opposite vertex of a triangle is called the **median**. Point *S* is the midpoint of $\overline{AC}$. Point *Q* is the midpoint of $\overline{BC}$. Point *R* is the midpoint of $\overline{AB}$.

The point where all three medians of a triangle intersect is called a **centroid**. Point *T* is the centroid of $\triangle ABC$.



Theorems

This unit of the core requires students to create proofs, formal or informal, to prove the following theorems. Example proofs will be provided; however, teachers will probably do tasks with students to complete these proofs in class.

Angle Sum Theorem: The sum of the measures of the angles of a triangle is 180°.

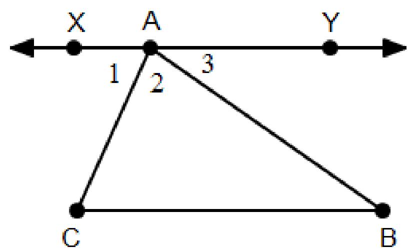
Isosceles Triangle Theorem: If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

Triangle Midsegment Theorem: A midsegment of a triangle is parallel to one side of the triangle, and its length is one-half the length of that side.

Theorem: The medians of a triangle meet at a point.

<u>Example 1:</u> Proof of the Angle Sum Theorem

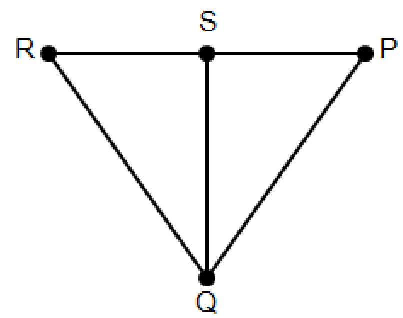
Given: $\triangle ABC$
Prove: $m\angle C + m\angle 2 + m\angle B = 180$



Draw $\overleftrightarrow{XY}$ through A so it is parallel to $\overline{CB}$. Because $\angle 1$ and $\angle CAY$ form a linear pair they are supplementary, so $m\angle 1 + m\angle CAY = 180°$. $\angle 2$ and $\angle 3$ form $\angle CAY$, so $m\angle 2 + m\angle 3 = m\angle CAY$. Using substitution, $m\angle 1 + m\angle 2 + m\angle 3 = 180°$. Because you drew $\overleftrightarrow{XY}$ parallel to $\overline{CB}$, we know $\angle 1 \cong \angle C$ and $\angle 3 \cong \angle B$ since they are alternate interior angles. Because the angles are congruent, their measures are equal. Therefore, using substitution again, we know $m\angle C + m\angle 2 + m\angle B = 180°$.

<u>Example 2:</u> Proof of the Isosceles Triangle Theorem

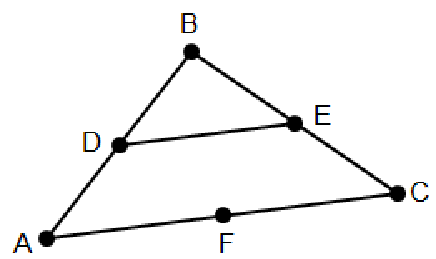
Given: $\triangle PQR$, $\overline{PQ} \cong \overline{RQ}$
Prove: $\angle P \cong \angle R$



Let S be the midpoint of $\overline{PR}$. Draw $\overline{SQ}$. Since S is the midpoint, $\overline{PS} \cong \overline{RS}$. $\overline{QS}$ is congruent to itself. Since were were given that $\overline{PQ} \cong \overline{RQ}$, we know that all 3 corresponding pairs of sides are congruent and we can say $\triangle PQS \cong \triangle RQS$ because of SSS congruency. Therefore, $\angle P \cong \angle R$ since they are corresponding angles of congruent triangles.

<u>Example 3:</u> Proof of the Triangle Midsegment Theorem

Given: Points *D*, *E*, and *F* are the midpoints of the sides of the triangle.
Prove: $\overline{DE} \parallel \overline{AC}$ and $\overline{DE} = \frac{\overline{AC}}{2}$



<u>Statement</u>	**<u>Reason</u>**
1. D and E are midpoints	1. Given
2. $\overline{DB} = \frac{1}{2}\overline{AB}$, $\overline{BE} = \frac{1}{2}\overline{BC}$	2. By definition of a midpoints
3. $\Delta BDE \sim \Delta BAC$	3. *B* is the center of dilation and scale factor is ½
4. $\overline{DE} \parallel \overline{AC}$	4. Dilations take lines to parallel lines
5. $\overline{DE} = \frac{\overline{AC}}{2}$	5. Scale factor is ½

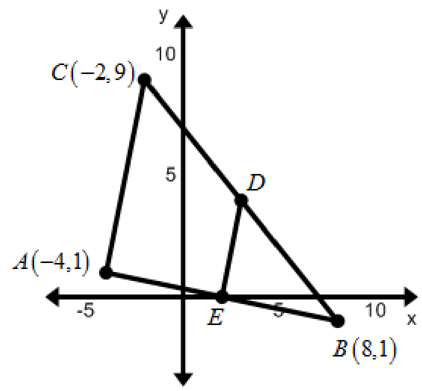
Example 4:

$\triangle ABC$ has vertices A(-4, 1), B(8, -1), and C(-2, 9). $\overline{DE}$ is a midsegment of $\triangle ABC$.

A. Find the coordinates of D and E.

B. Verify that $\overline{AC}$ is parallel to $\overline{DE}$.

C. Verify that $DE = \frac{1}{2}AC$.



Answers:

A.

$$D\left(\frac{-2+8}{2}, \frac{9+(-1)}{2}\right) = D\left(\frac{6}{2}, \frac{8}{2}\right) = D(3,4)$$

$$E\left(\frac{-4+8}{2}, \frac{1+(-1)}{2}\right) = E\left(\frac{4}{2}, \frac{0}{2}\right) = E(2,0)$$

Use the Midpoint Formula to find the midpoints of $\overline{AB}$ and $\overline{CB}$.

B.

$$\text{slope of } \overline{AC} = \frac{9-1}{-2-(-4)} = \frac{8}{2} = 4$$

$$\text{slope of } \overline{DE} = \frac{4-0}{3-2} = \frac{4}{1} = 4$$

If the slopes of $\overline{AC}$ and $\overline{DE}$ are equal, $\overline{AC} \parallel \overline{DE}$.

Because the slopes of $\overline{AC}$ and $\overline{DE}$ are equal, $\overline{AC} \parallel \overline{DE}$.

C.

$$AC = \sqrt{(-2-(-4))^2 + (9-1)^2} = \sqrt{(2)^2 + (8)^2} = \sqrt{4+64} = \sqrt{68}$$

$$DE = \sqrt{(3-2)^2 + (4-0)^2} = \sqrt{(1)^2 + (4)^2} = \sqrt{1+16} = \sqrt{17}$$

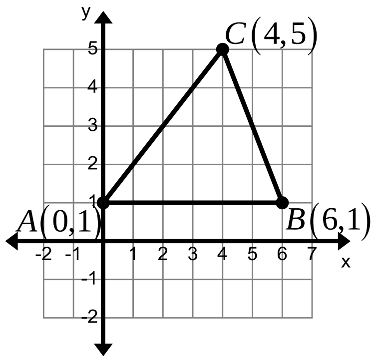
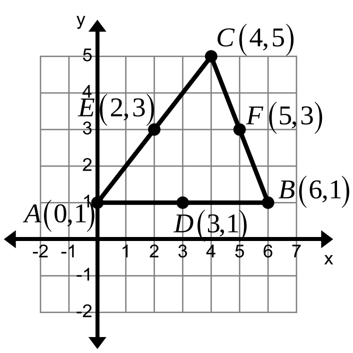
First, use the distance formula to find AC and DE.

$$\frac{DE}{AC} = \frac{\sqrt{17}}{\sqrt{68}} = \sqrt{\frac{17}{68}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Now show $DE = \frac{1}{2}AC$

Because $\frac{DE}{AC} = \frac{1}{2}$, $DE = \frac{1}{2}AC$.

Example 5: The medians of a triangle meet at a point.
Given a triangle with vertices at $A(0,1)$, $B(6,1)$, and $C(4,5)$, prove that the medians meet at a single point.

	Draw the triangle and label its vertices.
$D=\left(\frac{0+6}{2},\frac{1+1}{2}\right)=\left(\frac{6}{2},\frac{2}{2}\right)=(3,1)$ $E=\left(\frac{0+4}{2},\frac{1+5}{2}\right)=\left(\frac{4}{2},\frac{6}{2}\right)=(2,3)$ $F=\left(\frac{6+4}{2},\frac{1+5}{2}\right)=\left(\frac{10}{2},\frac{6}{2}\right)=(5,3)$ 	Find the midpoint of $\overline{AB}$ and label it D. Find the midpoint of $\overline{AC}$ and label it E. Find the midpoint of $\overline{CB}$ and label it F.
$m_{\overleftrightarrow{EB}}=\frac{3-1}{2-6}=\frac{2}{-4}=-\frac{1}{2}$ $y-3=-\frac{1}{2}(x-2)$ $m_{\overleftrightarrow{DC}}=\frac{1-5}{3-4}=\frac{-4}{-1}=4$ $y-1=4(x-3)$ $m_{\overleftrightarrow{FA}}=\frac{3-1}{5-0}=\frac{2}{5}$ $y-3=\frac{2}{5}(x-5)$	Write an equation for $\overleftrightarrow{EB}$, $\overleftrightarrow{DC}$, and $\overleftrightarrow{FA}$.

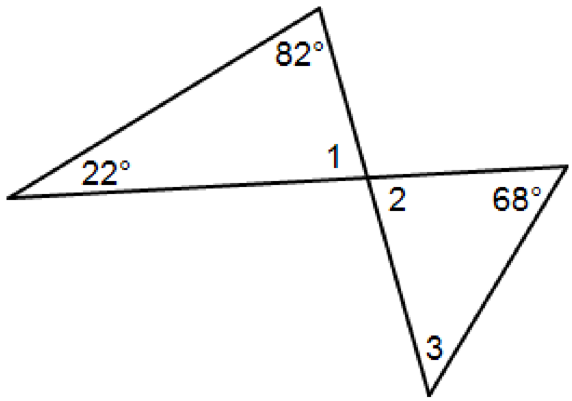
$y-3=-\frac{1}{2}(x-2)$ and $y-1=4(x-3)$ $y=-\frac{1}{2}x+4$ $\quad$ $y=4x-11$ $-\frac{1}{2}x+4=4x-11$ $-4.5x=-15$ $x=\frac{10}{3}$ $y=-\frac{1}{2}\left(\frac{10}{3}\right)+4$ $y=-\frac{5}{3}+4$ $y=\frac{7}{3}$	Using two of the equations, try to solve the system.
$y-3=\frac{2}{5}(x-5)$ $\frac{7}{3}-3=\frac{2}{5}\left(\frac{10}{3}-5\right)$ $-\frac{2}{3}=\frac{2}{5}\left(-\frac{5}{3}\right)$ $-\frac{2}{3}=-\frac{2}{3}$	See if the solution $\left(\frac{10}{3},\frac{7}{3}\right)$ works in the third equation.
All three medians of a triangle intersect in a single point.	

Practice Exercises A

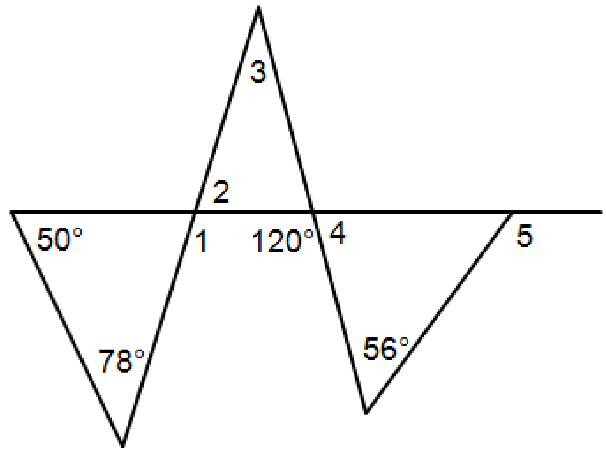
1. A base angle in an isosceles triangle measures 37°. Draw and label the triangle. What is the measure of the vertex angle?

Find the missing angle measures.

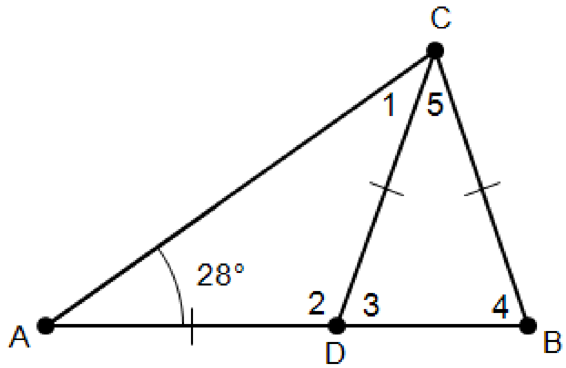
2.



3.

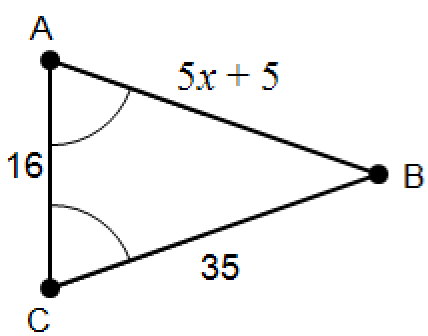


4.

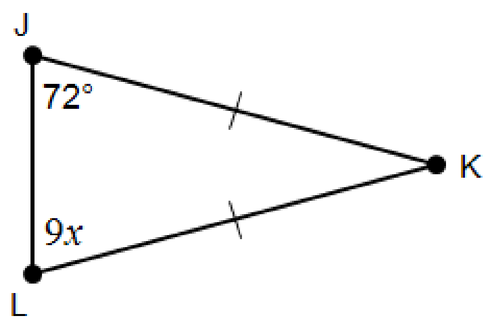


Find the value of *x*.

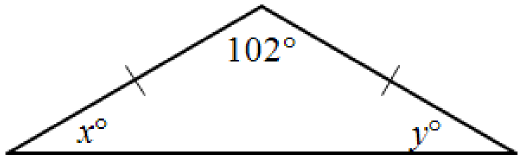
5.



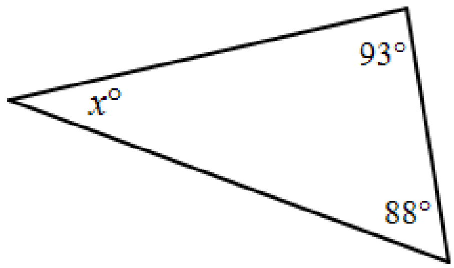
6.



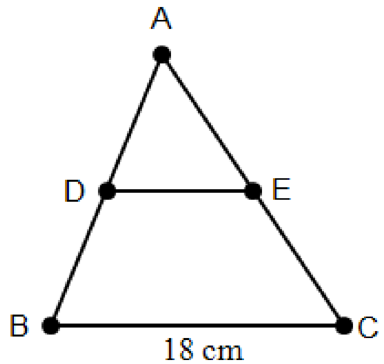
7. Find the values of both x and y



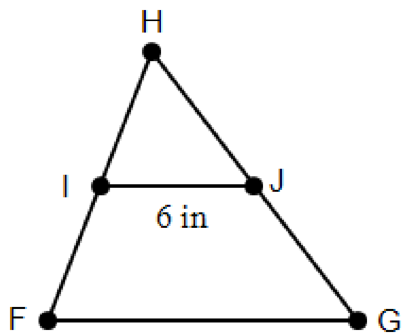
8. Can the given measurements be accurate? Why or why not?



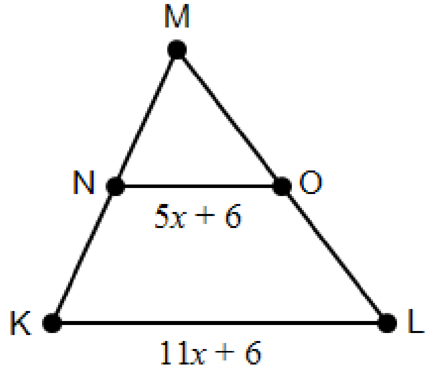
9. Find the length of $\overline{DE}$ given that D and E are midpoints.



10. Find the length of $\overline{FG}$ given that $\overline{IJ}$ is a midsegment of the triangle.



11. Solve for x given $\overline{NO}$ is a midsegment of the triangle.

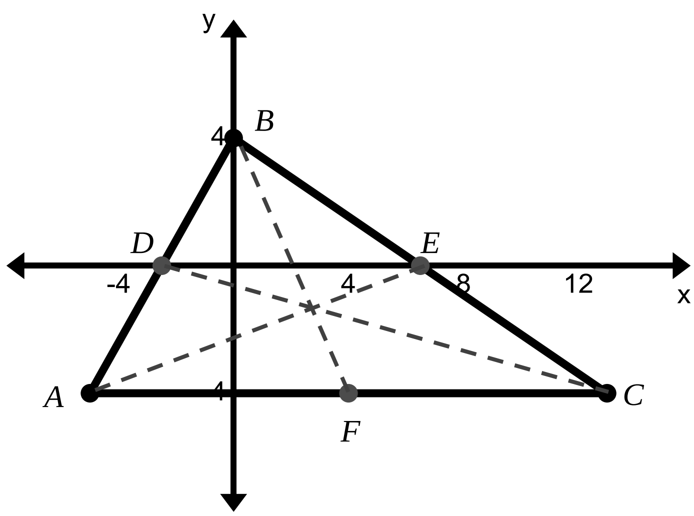


12. $\triangle ABC$ has vertices $A(-2,6)$, $B(-4,0)$, and $C(10,0)$. $\overline{DE}$ is a midsegment with D being the midpoint of of $\overline{AB}$ and E being the midpoint of $\overline{AC}$.

a. Find the coordinates of D and E.
b. Verify that $\overline{BC}$ is parallel to $\overline{DE}$.
c. Verify that $DE = \frac{1}{2}BC$.

13. ΔABC has vertices at $A(-5,-4)$, $B(0,4)$, and $C(13,-4)$.

a. Find the coordinates of D, E, and F, the midpoints of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.



b. Find the equations for two of the medians. Use a system of equations to find the location of the centroid (the point where the medians intersect).

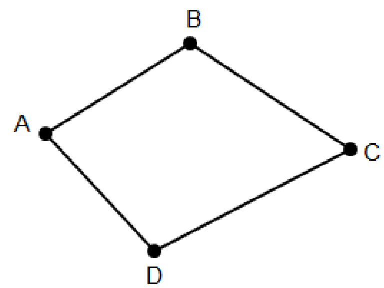
Unit 5 Cluster 2 (G.CO.11)
Prove Geometric Theorems about Parallelograms

Cluster 2: Prove Geometric Theorems

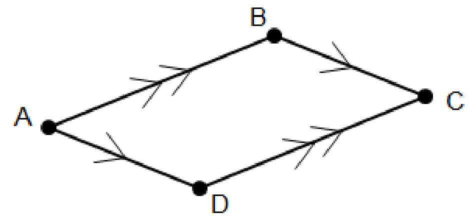
5.2.3 Prove theorems about parallelograms: opposite sides are congruent, opposite angles are congruent, the diagonals bisect each other, and rectangles are parallelograms with congruent diagonals.

VOCABULARY

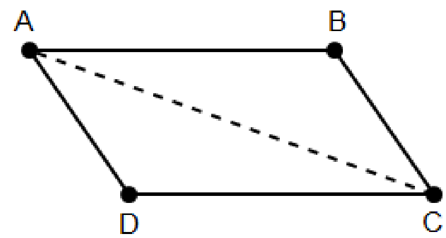
A polygon with four sides is called a **quadrilateral**.



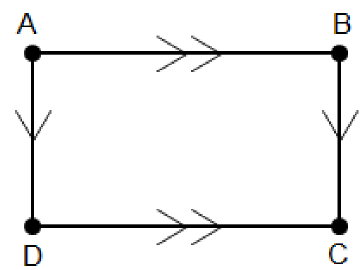
A quadrilateral with two pairs of parallel sides is called a **parallelogram**.



A segment that connects two nonconsecutive vertices is called a **diagonal**.



A parallelogram with four right angles is called a **rectangle**.



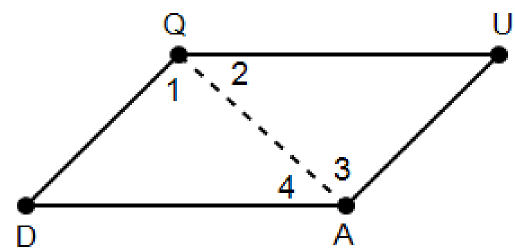
Theorems

This unit of the core requires students to create proofs, formal or informal, to prove the following theorems. Example proofs will be provided; however, teachers will probably do tasks with students to complete these proofs in class.

- If a quadrilateral is a parallelogram, then its opposite sides are congruent.
- If a quadrilateral is a parallelogram, then its opposite angles are congruent.
- If a quadrilateral is a parallelogram, then its diagonals bisect each other.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
- If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Exmaple 1: If a quadrilateral is a parallelogram, then its opposite sides are congruent.

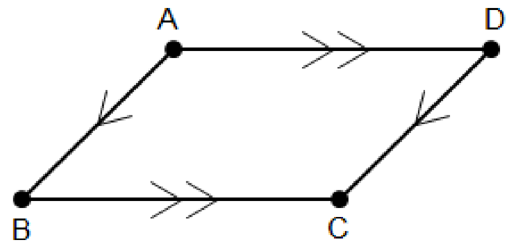
Given: QUAD is a parallelogram
Prove: $\overline{QU} \cong \overline{AD}$ and $\overline{DQ} \cong \overline{UA}$



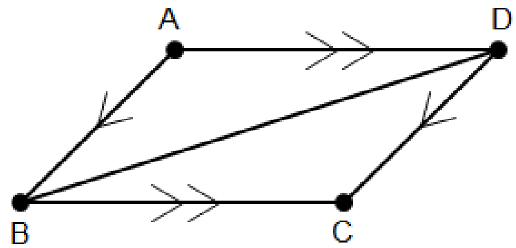
Statements	Reasons
1. QUAD is a parallelogram.	1. Given.
2. $\overline{QU} \parallel \overline{AD}$ and $\overline{DQ} \parallel \overline{UA}$	2. Definition of parallelogram.
3. $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$	3. Alternate interior angles are congruent.
4. $\overline{QA} \cong \overline{QA}$	4. Reflexive property.
5. $\triangle QUA \cong \triangle ADQ$	5. ASA congruence.
6. $\overline{QU} \cong \overline{AD}$ and $\overline{DQ} \cong \overline{UA}$	6. Corresponding parts of congruent triangles are congruent (CPCTC).

Example 2: If a quadrilateral is a parallelogram, then its opposite angles are congruent.

Given: ABCD is a parallelogram
Prove: $\angle A \cong \angle C$ and $\angle B \cong \angle D$



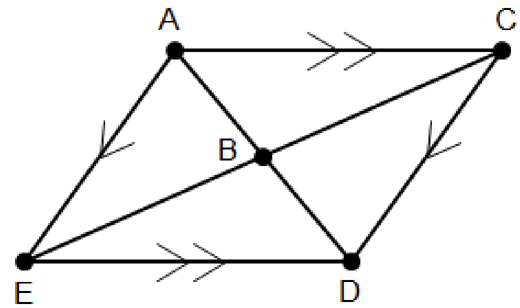
Since ABCD is a parallelogram, the opposite sides must be parallel.



Draw $\overline{BD}$ as a diagonal of the parallelogram. We know $\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{CD}$ because opposite sides of a parallelogram are congruent. $\overline{BD}$ is congruent to itself. This creates two congruent triangles by SSS: $\triangle ABD \cong \triangle CDB$. Because the triangles are congruent, the corresponding parts will be congruent. Therefore, $\angle A \cong \angle C$. The same logic can be used using $\overline{AC}$ as the diagonal to show $\angle B \cong \angle D$.

Example 3: If a quadrilateral is a parallelogram, then its diagonals bisect each other.

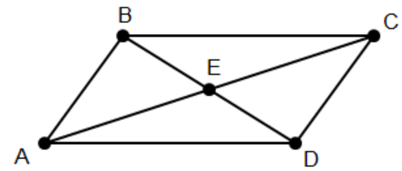
Given: ACDE is a parallelogram
Prove: $\overline{AB} \cong \overline{BD}$ and $\overline{EB} \cong \overline{BC}$



It is given that ACDE is a parallelogram. Since opposite sides of a parallelogram are congruent, $\overline{EA} \cong \overline{DC}$. By definition of a parallelogram, $\overline{EA} \parallel \overline{DC}$. $\angle AEB \cong \angle DCB$ and $\angle EAB \cong \angle CDB$ because alternate interior angles are congruent. $\triangle EBA \cong \triangle CBD$ by ASA. Since the triangles are congruent, their corresponding parts are also congruent. Therefore, $\overline{AB} \cong \overline{BD}$ and $\overline{EB} \cong \overline{BC}$. The diagonals of a parallelogram bisect each other.

Example 4: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Given: $\overline{AC}$ and $\overline{BD}$ bisect each other at E.
Prove: ABCD is a parallelogram



$\overline{AC}$ and $\overline{BD}$ bisect each other at E.

Given

$\overline{AE} \simeq \overline{EC}$

$\overline{BE} \simeq \overline{ED}$

Definition of a segment bisector

$\angle AEB \cong \angle CED$

Vertical Angles

$\triangle BEA \cong \triangle DEC$

SAS

$\angle BAE \cong \angle DCE$

Corresponding parts of congruent triangles are congruent (CPCTC).

$\overline{AB} \parallel \overline{CD}$

If alternate interior angles are congruent, then lines are ||

$\angle BEC \cong \angle DEA$

Vertical Angles

$\triangle AED \cong \triangle CEB$

SAS

$\angle ECB \cong \angle EAD$

Corresponding parts of congruent triangles are congruent (CPCTC).

$\overline{BC} \parallel \overline{AD}$

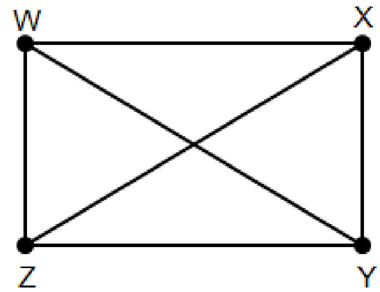
If alternate interior angles are congruent, then lines are ||

ABCD is a parallelogram.

Definition of parallelogram.

<u>Example 5:</u> If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Given: WXYZ is a parallelogram with $\overline{WY} \cong \overline{XZ}$
Prove: WXYZ is a rectangle



Statements	Reasons
1. $\overline{WY} \cong \overline{XZ}$	1. Given.
2. $\overline{XY} \cong \overline{ZW}$	2. Definition of a parallelogram.
3. $\overline{WX} \cong \overline{XW}$	3. Segment congruent to itself (reflexive property).
4. $\triangle WZX \cong \triangle XYW$	4. SSS congruence.
5. $\angle ZWX \cong \angle YXW$	5. Corresponding parts of congruent triangles are congruent.
6. $m\angle ZWX = m\angle YXW$	6. Definition of congruent.
7. $m\angle ZWX + m\angle WXZ + m\angle XZW = 180°$	7. Triangle sum theorem.
8. $\angle XZW \cong \angle ZXY$	8. Alternate interior angles are congruent.
9. $m\angle XZW = m\angle ZXY$	9. Definition of congruent.
10. $m\angle ZWX + m\angle WXZ + m\angle ZXY = 180°$	10. Substitution.
11. $m\angle WXZ + m\angle ZXY = m\angle YXW$	11. Angle addition postulate.
12. $m\angle ZWX + m\angle YXW = 180°$	12. Substitution.
13. $\angle ZWX$ and $\angle YXW$ are right angles	13. If two angles are congruent and supplementary, each angle is a right angle.
14. $\angle WZY$ and $\angle XYZ$ are right angles	14. Opposite angles of a parallelogram are congruent.
15. WXYZ is a rectangle	15. Definition of a rectangle.

Practice Exercises A

1. Complete the flow proof by filling in the blanks for the theorem: If a quadrilateral is a parallelogram, then its opposite angles are congruent.

Given: $JKLM$ is a parallelogram
Prove: $\angle J \cong \angle L$ and $\angle K \cong \angle M$

$JKLM$ is a parallelogram
*a.*______________

$\angle J$ and $\angle K$ are consecutive $\angle$s.
Def. of consecutive $\angle$s

$\angle K$ and $\angle L$ are consecutive $\angle$s.
*b.*______________________

*c.*__________________
Def. of consecutive $\angle$s

*d.*____________________
Consecutive $\angle$s are supplementary

$\angle K$ and $\angle L$ are supplementary
Consecutive $\angle$s are supplementary

$\angle L$ and $\angle M$ are supplementary
Consecutive $\angle$s are supplementary

*e.*______________
Supplements of the same $\angle$ are $\cong$

$\angle K \cong \angle M$
f._________________
____

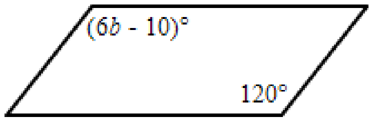
2. Complete the two column proof by filling in the blanks for the theorem: If a parallelogram is a rectangle, then its diagonals are congruent.

Given: ABCD is a rectangle
Prove: $\overline{AC} \cong \overline{BD}$

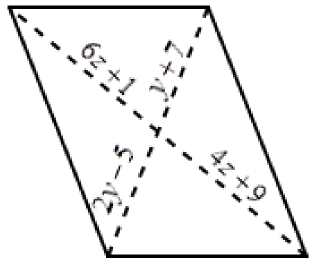
Statements	Reasons
1. ABCD is a rectangle	1. a. ______________________
2. $ABCD$ is a $\square$	2. b. ______________________
3. $\overline{BC} \cong \overline{CB}$	3. c.______________________
4. $\angle ABC$ and $\angle$DCB are right $\angle$s.	4. d.______________________
5. $\angle ABC \cong \angle DCB$	5. e.________________________
6. f. _______________________	6. Opposite sides of a parallelogram are congruent.
7. g.________________________	7. SAS
8. $\overline{AC} \cong \overline{BD}$	8. h._________________________

3. Find the values(s) of the variables(s) in each parallelogram.

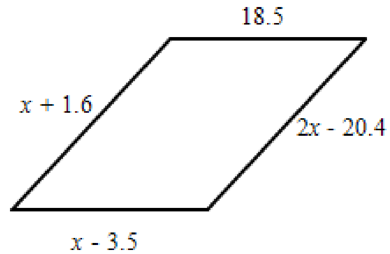
a.



b.

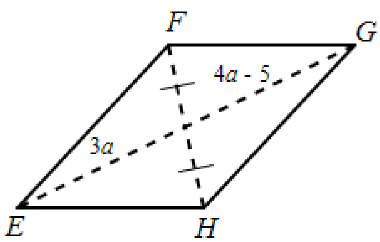


c.



4. For what values of *a* and *b* must *EFGH* be a parallelogram?

a.



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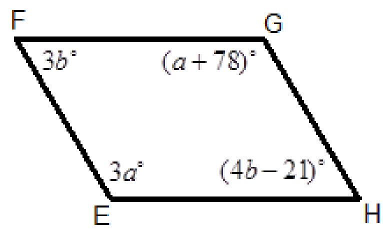

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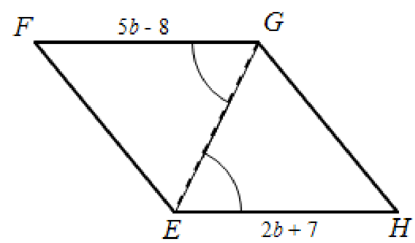


5. *QRST* is a rectangle. Find the value of *x* and the length of each diagonal.

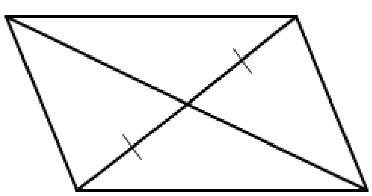
a. $QS = x$ and $RT = 2x - 4$

b. $QS = 7x - 2$ and $RT = 4x + 3$

c. $QS = 5x - 8$ and $RT = 2x + 1$

6. Is the given information enough to prove that the quadrilateral is a parallelogram? Explain why or why not.

a.



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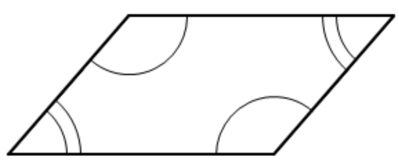
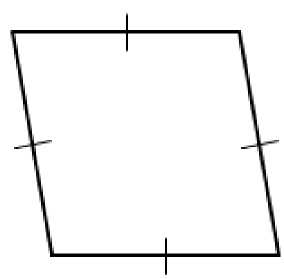


c.



Unit 5 Cluster 3 (G.SRT.4, G.SRT.5)
Prove Theorems Involving Similarity

Cluster 3: Prove theorems involving similarity

- 5.3.1 Prove theorems about triangles: a line parallel to one side divides the other 2 proportionally, the Pythagorean Theorem
- 5.3.2 Use congruence and similarity criteria to solve problems and prove relationships

Theorems

This unit of the core requires students to create proofs, formal or informal, to prove the following theorems. Example proofs will be provided; however, teachers will probably do tasks with students to complete these proofs in class.

Triangle Proportionality Theorem: If a line is parallel to one side of a triangle, then it divides the other two sides proportionally.

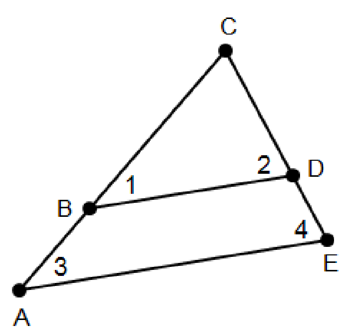
Converse of the Triangle Proportionality Theorem: If a line intersects two sides of a triangle proportionally, then that line is parallel to one side of the triangle.

Pythagorean Theorem: If a triangle is a right triangle with hypotenuse *c*, then $a^2 + b^2 = c^2$.

Proof of the Triangle Proportionality Theorem: A line parallel to one side of a triangle divides the other two sides proportionally.

Given: $\overline{BD} \parallel \overline{AE}$

Prove: $\frac{BA}{CB} = \frac{DE}{CD}$



Since $\overline{BD} \parallel \overline{AE}, \angle 3 \cong \angle 1$ and $\angle 4 \cong \angle 2$ because they are corresponding angles. Then by AA Similarity, $\triangle ACE \sim \triangle BCD$. By definition of similar polygons, $\frac{CA}{CB} = \frac{CE}{CD}$. From the Segment Addition Postulate, $CA = BA + CB$ and $CE = DE + CD$. Substituting for CA and CE in the ratio, we get the following proportion.

$$\frac{BA+CB}{CB}=\frac{DE+CD}{CD}$$

$$\frac{CB}{CB}+\frac{BA}{CB}=\frac{CD}{CD}+\frac{DE}{CD}$$ Rewrite as a sum.

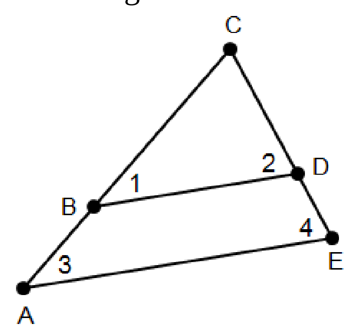
$$\frac{BA}{CB}+1=\frac{DE}{CD}+1$$ $\frac{CB}{CB}=1$ and $\frac{CD}{CD}=1$

$$\frac{BA}{CB}=\frac{DE}{CD}$$ Subtract 1 from each side

<u>Proof of the Converse of the Triangle Proportionality Theorem:</u> If a line intersects two sides of a triangle proportionally, then that line is parallel to one side of the triangle.

Given: $\frac{BA}{CB}=\frac{DE}{CD}$

Prove: $\overline{BD} \parallel \overline{AE}$

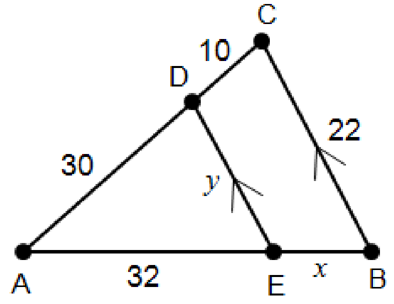


Statements	Reasons
1. $\frac{BA}{CB}=\frac{DE}{CD}$	1. Given.
2. $1+\frac{BA}{CB}=1+\frac{DE}{CD}$	2. Addition property of equality.
3. $\frac{CB}{CB}+\frac{BA}{CB}=\frac{CD}{CD}+\frac{DE}{CD}$	3. Substitution for 1.
4. $\frac{CB+BA}{CB}=\frac{CD+DE}{CD}$	4. Common denominator.
5. CA = CB + BA and CE = CD + DE	5. Segment addition postulate.
6. $\frac{CA}{CB}=\frac{CE}{CD}$	6. Substitution; sides are proportional.
7. $\angle C \cong \angle C$	7. Reflexive property of congruence.
8. $\Delta ACE \sim \Delta BCD$	8. SAS Similarity.
9. $\angle CBD \cong \angle CAB$	9. By definition of similar triangles.
10. $\overline{BD} \parallel \overline{AE}$	10. $\angle CBD$ and $\angle CAB$ are corresponding angles. Since they are congruent, the segments $\overline{BD}$ and $\overline{AE}$ are parallel.

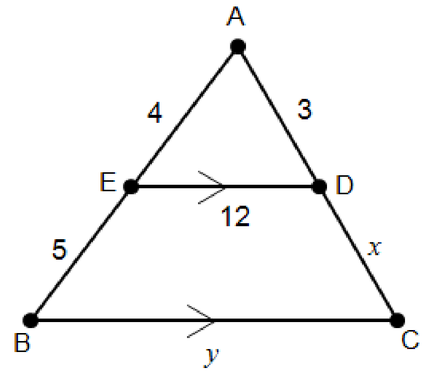
Practice Exercises A

Solve for x and y given $\triangle ABC \sim \triangle AED$.

1.

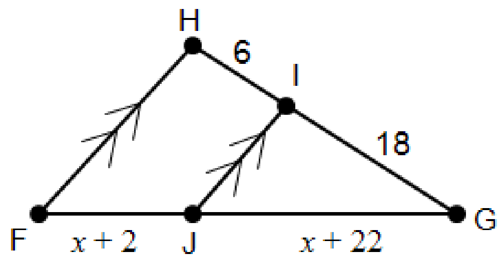


2.

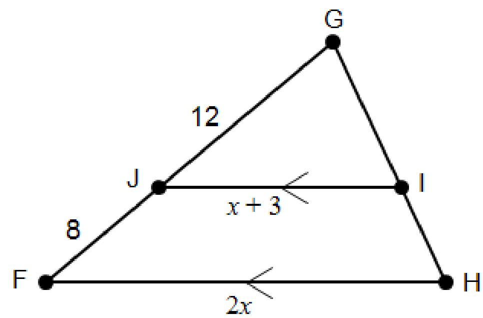


Solve for x.

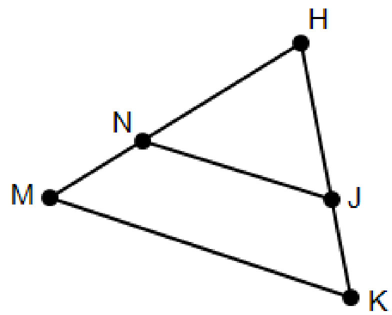
3.



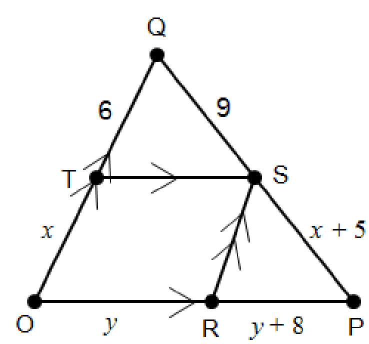
4.



5. In $\triangle HKM$, HM = 15, HN = 10, and $\overline{HJ}$ is twice the length of $\overline{JK}$. Determine whether $\overline{NJ} \parallel \overline{MK}$. Explain.

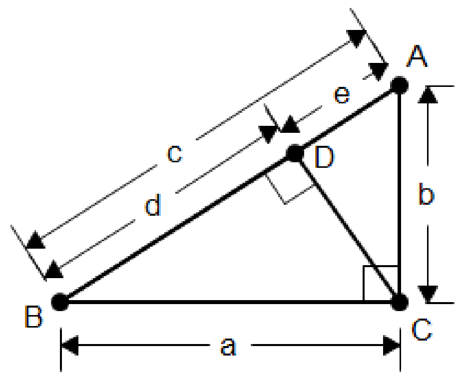


6. Find TO, SP, OR, and RP.



Proof of the Pythagorean Theorem: If a triangle is a right triangle with hypotenuse *c*, then $a^2 + b^2 = c^2$.

Given: $\triangle ABC$ is a right triangle with hypotenuse *c*
Prove: $a^2 + b^2 = c^2$



Draw $\overline{CD}$ so it is perpendicular to $\overline{AB}$. This creates three right triangles, $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$. All three of these triangles are similar by AA similarity. $\triangle ABC \sim \triangle CBD$ because they both have a right angle and they have $\angle B$ in common. $\triangle ABC \sim \triangle ACD$ because they both have a right angle and they have $\angle A$ in common. $\triangle ACD \sim \triangle CBD$ because they are both similar to $\triangle ABC$.

Since we know the triangles are similar, we also know the ratios of the sides are all the same. Start with $\triangle ABC$ and $\triangle CBD$.

$$\frac{BC}{AB} = \frac{BD}{CB} \Rightarrow \frac{a}{c} = \frac{d}{a} \Rightarrow a^2 = cd$$

Now use the same logic with $\triangle ABC$ and $\triangle ACD$.

$$\frac{AC}{AB} = \frac{AD}{AC} \Rightarrow \frac{b}{c} = \frac{e}{b} \Rightarrow b^2 = ce$$

Next, add the two equations together and factor out *c*.

$$a^2 + b^2 = cd + ce$$
$$a^2 + b^2 = c(d + e)$$

Looking at the original picture we see $d + e = c$, so we can substitute.

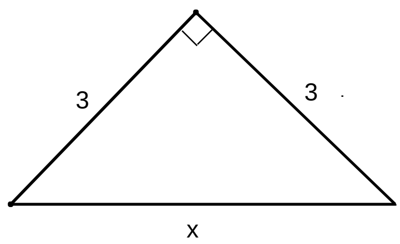
$$a^2 + b^2 = c(c)$$
$$a^2 + b^2 = c^2$$

Therefore, given a right triangle with hypotenuse *c*, $a^2 + b^2 = c^2$.

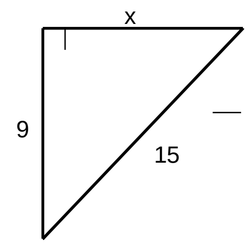
Practice Exercises B

Solve for *x*.

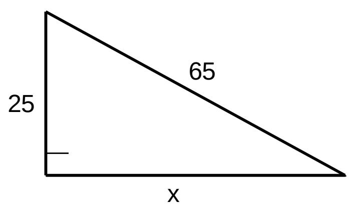
1.



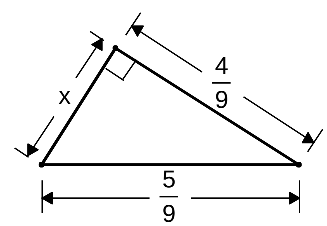
2.



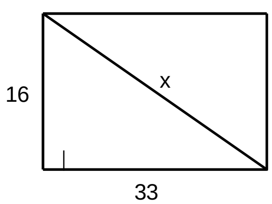
3.



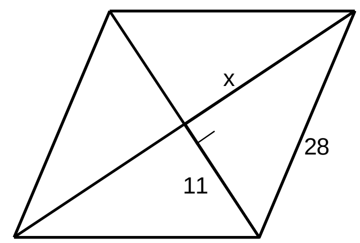
4.



5.



6.



Unit 5 Cluster 4 (G.GPE.6)
Coordinte Proofs

Cluster 4: Use coordinates to prove simple geometric theorems algebraically.

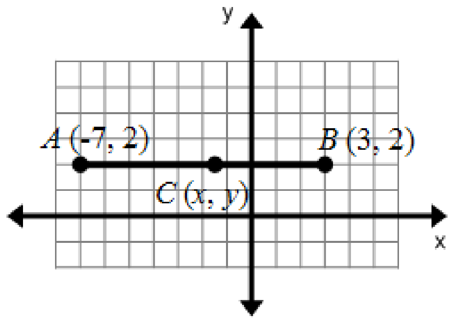
5.4.1 Find the point on a directed line segment between two given points that partitions that segment in a given ratio

Concept

For any segment with endpoints A and B, a point C between A and B will partition the segment into a given ratio.

Example 1: Horizontal Line Segment

Find the coordinates of *C* so that the ratio $\frac{AC}{CB} = 4$.

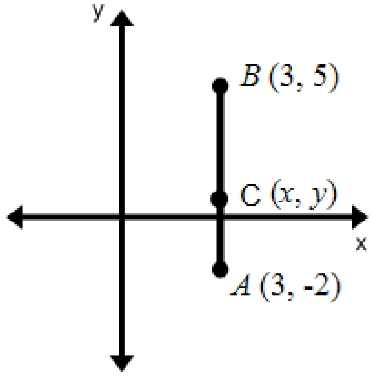


Answer:

$C(x,y) = C(x,2)$	Because $\overline{AB}$ is a horizontal line, the *y*-coordinate of *C* must be 2.
$\lvert 3-(-7)\rvert = \lvert 10\rvert = 10$	Find the horizontal distance from point *A* to point *B*.
$AC = AB - CB$	Use the segment addition postulate $AC + CB = AB$ and solve for *AC*.
$\frac{AB - CB}{CB} = 4$	Substitute $AC = AB - CB$ into $\frac{AC}{CB} = 4$.
$AB - CB = 4CB$ $AB = 5CB$	Multiply each side by *CB*. Add CB to each side.
$10 = 5CB$ $2 = CB$	Since $AB = 10$, substitute 10 in for *AB* and solve for *CB*.
$C(x,y) = C(1,2)$	Since $CB = 2$, the *x*-coordinate of the point *C* must be $3 - 2 = 1$.

Example 2: Vertical Line Segment

Find the coordinates of C so that the ratio $\frac{AC}{CB}=\frac{2}{5}$.

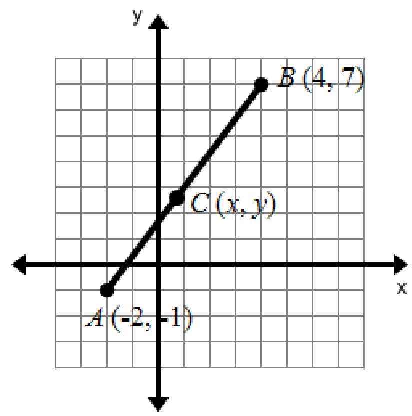


Answer:

$C(x,y)=C(3,y)$	Because $\overline{AB}$ is a vertical line, the x-coordinate of C must be 3.
$\lvert 5-(-2)\rvert=\lvert 7\rvert=7$	Find the vertical distance from point A to point B.
$AC=AB-CB$	Use the segment addition postulate $AC+CB=AB$ and solve for AC.
$\frac{AB-CB}{CB}=\frac{2}{5}$	Substitute $AC=AB-CB$ into $\frac{AC}{CB}=\frac{2}{5}$.
$AB-CB=\frac{2}{5}CB$ $5(AB-CB)=2CB$ $5AB-5CB=2CB$ $5AB=7CB$	Multiply each side by CB. Multiply each side by 5 and simplify. Add $5CB$ to each side.
$5(7)=7CB$ $35=7CB$ $5=CB$	Since $AB=7$, substitute 7 in for AB and solve for CB.
$C(x,y)=C(3,0)$	Since $CB=5$, the y-coordinate of the point C must be $5-5=0$.

Example 3: Positive Slope Line Segment

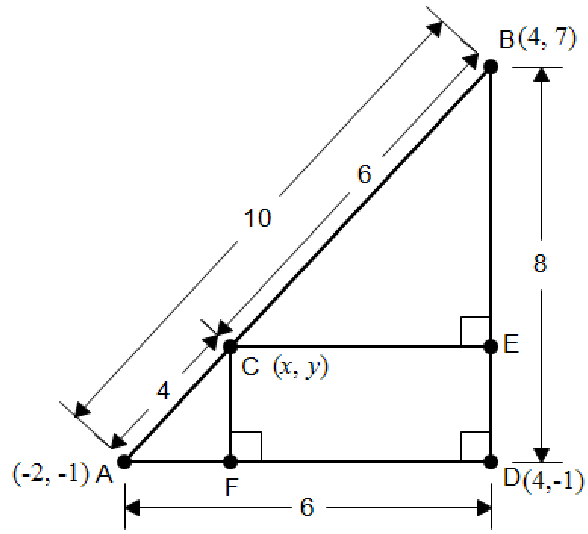
Find the coordinates of C such that $\frac{AC}{CB}=\frac{2}{3}$



Answer:

$AB=\sqrt{\left(4-(-2)\right)^2+\left(7-(-1)\right)^2}$ $AB=\sqrt{6^2+8^2}$ $AB=\sqrt{36+64}$ $AB=\sqrt{100}$ $AB=10$	Find the length of AB using the distance formula.
$AC=AB-CB$	Use the segment addition postulate $AC+CB=AB$ and solve for AC.
$\frac{AB-CB}{CB}=\frac{2}{3}$	Substitute $AC=AB-CB$ into $\frac{AC}{CB}=\frac{2}{3}$.
$AB-CB=\frac{2}{3}CB$ $3(AB-CB)=2CB$ $3AB-3CB=2CB$	Multiply each side by CB. Multiply each side by 3 then simplify.
$3AB=5CB$ $3(10)=5CB$ $30=5CB$ $6=CB$	Add $3CB$ to each side. Since $AB=10$, substitute 10 in for AB and solve for CB.
$AC+CB=AB$ $AC+6=10$ $AC=4$	We can find AC by using the segment addition postulate and substituting in the known values.

To find the coordinates of point C, you will need to draw a right triangle with $\overline{AB}$ as the hypotenuse. Then draw a line from point C perpendicular to each leg of the right triangle. The two small triangles created within the larger triangle are similar to each other and the larger triangle by AA similarity. Therefore the ratio of their sides is equal.



You need the coordinates of point F to find the x-coordinate of point C. Use the ratios to find AF.

$$\frac{AC}{AB} = \frac{AF}{AD}$$

$$\frac{4}{10} = \frac{AF}{6}$$

$$10AF = 24$$

$$AF = 2.4$$

$$F = (-2+2.4, -1) = (0.4, -1)$$

Point F will have the same y-coordinate as point A. The x-coordinate will be the same as A plus 2.4 because AF is 2.4.

You need the coordinates of point E to find the y-coordinate of point C. Use the ratios to find DE.

$$\frac{AC}{AB} = \frac{DE}{BD}$$

$$\frac{4}{10} = \frac{DE}{8}$$

$$10DE = 32$$

$$DE = 3.2$$

$$E = (4, -1+3.2) = (4, 2.2)$$

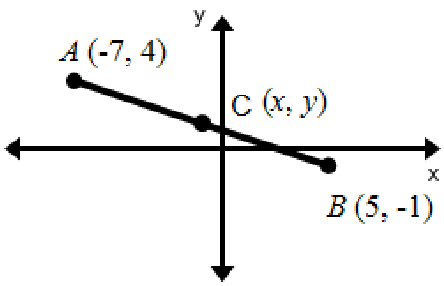
Point E will have the same x-coordinate as point D. The y-coordinate will be the same as D plus 3.2 because DE is 3.2.

$$C(0.4, 2.2)$$

The x-coordinate of F is 0.4 and the y-coordinate of E is 2.2.

Example 4: Negative Slope Line Segment

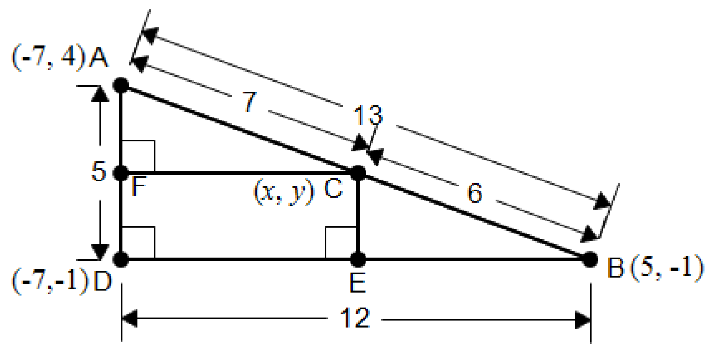
Find the coordinates of C such that $\frac{AC}{CB}=\frac{7}{6}$



Answer:

$AB=\sqrt{(-7-5)^2+(4-(-1))^2}$ $AB=\sqrt{(-12)^2+5^2}$ $AB=\sqrt{144+25}$ $AB=\sqrt{169}$ $AB=13$	Find the length of AB using the distance formula.
$AC=AB-CB$	Use the segment addition postulate $AC+CB=AB$ and solve for *AC*.
$\frac{AB-CB}{CB}=\frac{7}{6}$	Substitute $AC=AB-CB$ into $\frac{AC}{CB}=\frac{7}{6}$.
$AB-CB=\frac{7}{6}CB$ $6(AB-CB)=7CB$ $6AB-6CB=7CB$	Multiply by each side by *CB.* Multiply each side by 6 then simplify.
$6AB=13CB$ $6(13)=13CB$ $78=13CB$ $6=CB$	Add 6*CB* to each side. Since $AB=13$, substitute 13 in for *AB* and solve for *CB*.
$AC+CB=AB$ $AC+6=13$ $AC=7$	We can find *AC* by using the segment addition postulate and substituting in the known values.

To find the coordinates of point *C*, you will need to draw a right triangle with $\overline{AB}$ as the hypotenuse. Then draw a line from point C perpendicular to each leg of the right triangle. The two small triangles created within the larger triangle are similar to each other and the larger triangle by AA similarity. Therefore the ratio of their sides is equal.



$$\frac{AC}{AB}=\frac{AF}{AD}$$

You need the coordinates of point *F* to find the *y*-coordinate of point *C*. Use the ratios to find *AF*.

$$\frac{7}{13}=\frac{AF}{5}$$

$$13AF=35$$

$$AF=\frac{35}{13}\approx 2.69$$

$$F=\left(-7,4-\frac{35}{13}\right)=\left(-7,\frac{17}{13}\right)\approx(-7,1.31)$$

Point *F* will have the same *x*-coordinate as point *A*. The *y*-coordinate will be the same as *A* minus $\frac{35}{13}$ because *AF* is $\frac{35}{13}$.

$$\frac{AC}{AB}=\frac{DE}{BD}$$

$$\frac{7}{13}=\frac{DE}{12}$$

You need the coordinates of point *E* to find the *x*-coordinate of point *C*. Use the ratios to find *DE*.

$$13DE=84$$

$$DE=\frac{84}{13}\approx 6.46$$

$$E=\left(-7+\frac{84}{13},-1\right)=\left(-\frac{7}{13},-1\right)\approx(-0.54,-1)$$

Point *E* will have the same *y*-coordinate as point *D*. The *x*-coordinate will be the same as *D* plus $\frac{84}{13}$ because *DE* is $\frac{84}{13}$.

$$C\left(-\frac{7}{13},\frac{17}{13}\right)\approx C(-0.54,1.31)$$

The *x*-coordinate of F is $-\frac{7}{13}$ and the *y*-coordinate of E is $\frac{17}{13}$.

Practice Exercises A

1. C is between A and B with A (3, -5) and B (3, 7). Find the coordinates of C such that $\frac{AC}{CB} = 3$.
2. C is between A and B with $A(-3,-1)$ and $B(6,-1)$. Find the coordinates of C such that $\frac{AC}{CB} = \frac{5}{4}$.
3. C is between A and B with $A(2,-4)$ and $B(2,4)$. Find the coordinates of C such that $\frac{AC}{CB} = \frac{1}{3}$.
4. C is between A and B with $A(-2,5)$ and $B(6,5)$. Find the coordinates of C such that $\frac{AC}{CB} = \frac{5}{3}$.
5. C is between A and B with A (-5, -3) and B (3, 3). Find the coordinates of C such that $\frac{AC}{CB} = \frac{3}{7}$.

Unit 5 Cluster 5 (G.SRT.6, G.SRT.7, G.SRT.8)
Right Triangle Trigonometry

Cluster 5: Defining trigonometric ratios and solving problems

- 5.5.1 Similarity of triangles leads to the trigonometric ratios of the acute angles.
- 5.5.2 Understand the relationship between the sine and cosine of complimentary angles.
- 5.5.3 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems

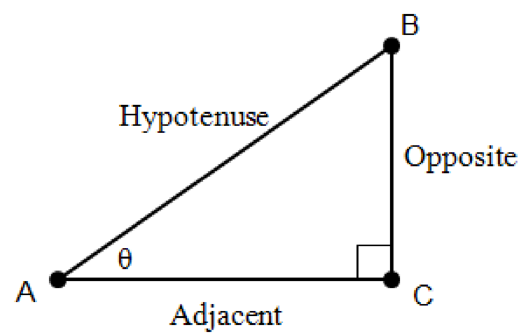
Two figures that have congruent angles and proportional sides are similar. The requirements for proving that two right triangles are similar are less than the requirements needed to prove non-right triangles similar. It is already known that both triangles have one right angle. Therefore, two right triangles are similar if one acute angle is congruent to one acute angle in the other right triangle. In a right triangle, the ratio between the side lengths is a function of an acute angle ($0°$ to $90°$). There are six distinct trigonometric ratios that are functions of an acute angle in a right triangle. Right triangles that are similar will have the same trigonometric ratios.

VOCABULARY

The side opposite the right angle is the called the **hypotenuse**.

The side that meets the hypotenuse to form the angle θ is called the **adjacent** side.

The side that is opposite the angle θ is called the **opposite** side.



Trigonometric Ratios

Let θ be an acute angle in the right $\triangle ABC$ as shown in the figure above. Then,

sine $(\theta) = \sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

cosecant $(\theta) = \csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

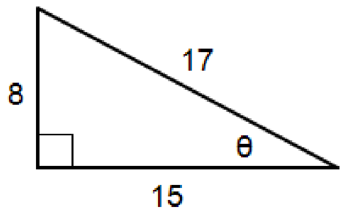
cosine $(\theta) = \cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

secant $(\theta) = \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

tangent $(\theta) = \tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

cotangent $(\theta) = \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\text{adjacent}}{\text{opposite}}$

Example: Given the figure below, identify all six trigonometric ratios of the angle θ.



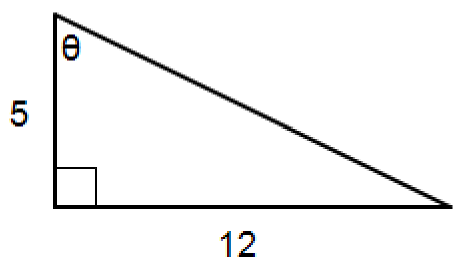
Answer:
The opposite side is 8. The adjacent side is 15. The hypotenuse is 17. Therefore,

$$\sin\theta = \frac{8}{17} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{8}{17}} = \frac{17}{8}$$

$$\cos\theta = \frac{15}{17} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{15}{17}} = \frac{17}{15}$$

$$\tan\theta = \frac{8}{15} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{8}{15}} = \frac{15}{8}$$

Example: Given the figure below, identify all six trigonometric ratios of the angle θ.



Answer:
The opposite side is 12. The adjacent side is 5. The hypotenuse is not known. Use the Pythagorean Theorem to find the missing hypotenuse and then calculate the six trigonometric ratios.

$$5^2 + 12^2 = c^2$$
$$25 + 144 = c^2$$
$$169 = c^2$$
$$13 = c$$

The hypotenuse is 13.

$$\sin\theta = \frac{12}{13} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

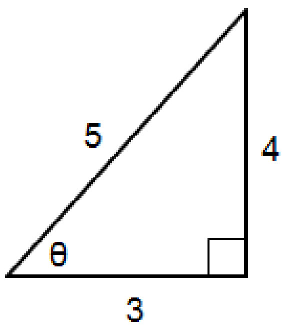
$$\cos\theta = \frac{5}{13} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$$

$$\tan\theta = \frac{12}{5} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{12}{5}} = \frac{5}{12}$$

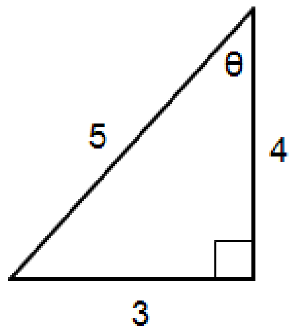
Practice Exercises A

Given the figures below, identify all six trigonometric ratios of the angle θ.

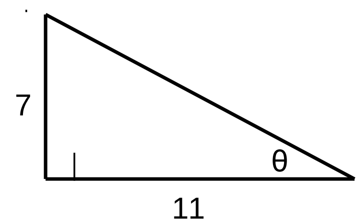
1.



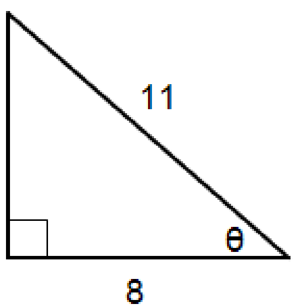
2.



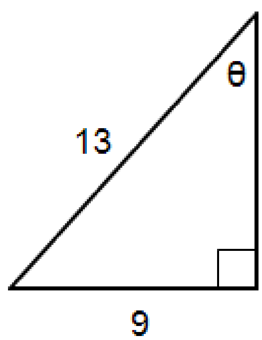
3.



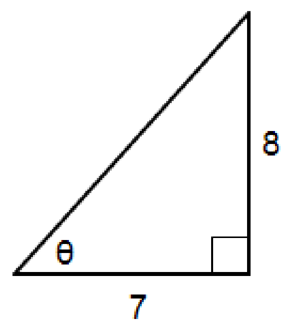
4.



5.



6.



Use a calculator to find each value.

7. $\sin(9°)$ 8. $\cos(37°)$ 9. $\tan(48°)$

10. $\cos(55°)$ 11. $\tan(72°)$ 12. $\sin(23°)$

When solving equations, you use the inverse operation to find the value of the variable. In trigonometry, you can find the measure of the angle by using the inverse of sine, cosine, or tangent.

Equation	Inverse Equation	Meaning
$\sin(\theta)=\frac{x}{y}$	$\sin^{-1}\left(\frac{x}{y}\right)=\theta$	The inverse, or arcsine, of $\frac{x}{y}$ is equal to the angle θ.
$\cos(\theta)=\frac{x}{y}$	$\cos^{-1}\left(\frac{x}{y}\right)=\theta$	The inverse, or arccosine, of $\frac{x}{y}$ is equal to the angle θ.
$\tan(\theta)=\frac{x}{y}$	$\tan^{-1}\left(\frac{x}{y}\right)=\theta$	The inverse, or arctangent, of $\frac{x}{y}$ is equal to the angle θ.

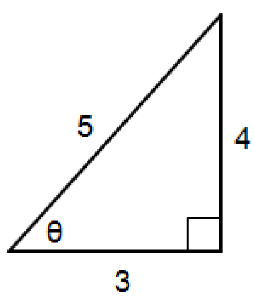
Practice Exercises B

Use a calculator to find each value.

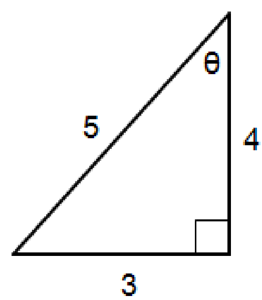
1. $\sin^{-1}(0.5)$
2. $\cos^{-1}(0.86)$
3. $\tan^{-1}(6)$
4. $\tan^{-1}(1)$
5. $\sin^{-1}(0.75)$
6. $\cos^{-1}(0.33)$

Given the figures below, find the measure of the angle θ.

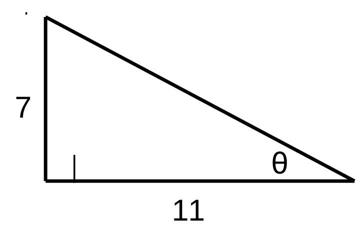
7.



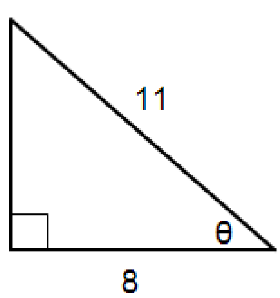
8.



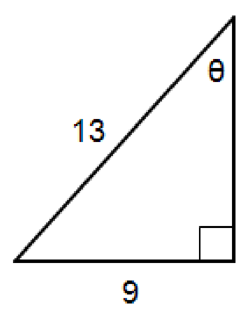
9.



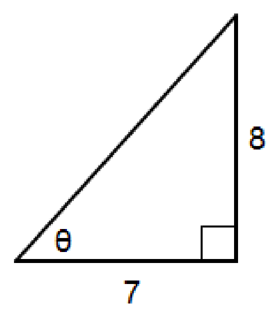
10.



11.

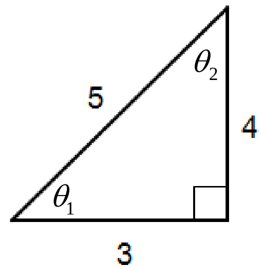


12.

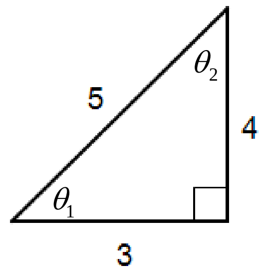


Relationship Between Complementary Angles (Cofunction Identities)

$$\cos(90° - \theta_1) = \sin(\theta_1)$$
$$\cos(\theta_2) = \sin(\theta_1)$$
$$\frac{4}{5} = \frac{4}{5}$$



$$\sin(90° - \theta_1) = \cos(\theta_1)$$
$$\sin(\theta_2) = \cos(\theta_1)$$
$$\frac{3}{5} = \frac{3}{5}$$



Practice Exercises C

Use the relationship of complementary angles to find the missing angle.

1. If $\sin(30°) = \frac{1}{2}$, then $\cos(\_\_\_\_°) = \frac{1}{2}$

2. If $\cos(0°) = 1$, then $\sin(\_\_\_\_°) = 1$

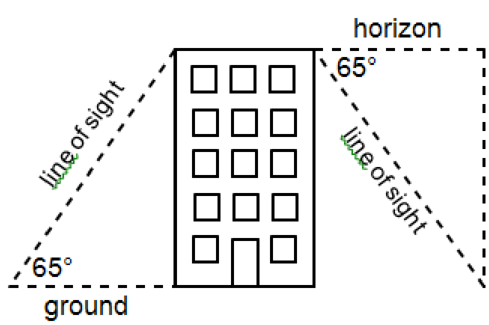
3. If $\cos(23°) = 0.92$, then $\sin(\_\_\_\_°) = 0.92$

4. If $\sin(75°) = 0.97$, then $\cos(\_\_\_\_°) = 0.97$

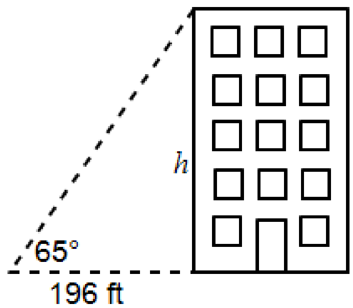
VOCABULARY

An **angle of elevation** is the angle made with the ground and your line of sight to an object above you.

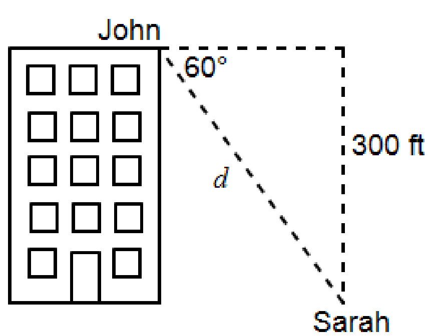
An **angle of depression** is the angle from the horizon and your line of sight to an object below you.



<u>Example 1:</u>
You are standing 196 feet from the base of an office building in downtown Salt Lake City. The angle of elevation to the top of the building is $65°$. Find the height, *h*, of the building.

	Draw a picture of the situation and label what you know.
$\tan(65°) = \frac{h}{196}$	You know the adjacent side and want the opposite side. Use the tangent ratio to help you set up the problem.
$\tan(65°) = \frac{h}{196}$ $196 \cdot \tan(65°) = h$ $420.32 = h$	Solve for *h*.
The height of the building is approximately 420.32 feet high.	

Example 2:
John is standing on the roof of a building that is 300 feet tall and sees Sarah standing on the ground. If the angle of depression is $60°$ how far away is Sarah from John?

	Draw a picture of the situation and label what you know.
$\sin(60°) = \frac{300}{d}$	You know the opposite side and want the hypotenuse. Use the sine ratio to help you set up the problem.

$\sin(60°)=\frac{300}{d}$ $d\cdot\sin(60°)=300$ $d=\frac{300}{\sin(60°)}$ $d=346.41$	Solve for *d*.
Sarah is approximately 346.41 feet away from John.	

Practice Exercises D

Solve each problem.

1. The angle of depression from the top of a lighthouse 150 feet above the surface of the water to a boat is $13°$. How far is the boat from the lighthouse?

2. A guy wire connects the top of an antenna to a point on the level ground 7 feet from the base of the antenna. The angle of elevation formed by this wire is $75°$. What are the length of the wire and the height of the antenna?

3. A private jet is taking off from Telluride, Colorado. The runway is 46,725 feet from the base of the mountain. The plane needs to clear the top of Mount Sneffels, which is 14,150 feet high, by 100 feet. What angle should the plane maintain during takeoff?

4. A person is 75 feet from the base of a barn. The angle of elevation from the level ground to the top of the barn is $60°$. How tall is the barn?

5. From the top of a building 250 feet high, a man observes a car moving toward him. If the angle of depression of the car changes from $18°$ to $37°$, how far does the car travel while the man is observing it?

6. A rocket is launched from ground level. A person standing 84 feet from the launch site observes that the angle of elevation is $71°$ at the rocket's highest point. How high did the rocket reach?

7. A hot-air balloon is 700 feet above the ground. The angle of depression from the balloon to an observer is $5°$. How far is the observer from the hot-air balloon?

8. If a wheelchair access ramp has to have an angle of elevation of no more than $4.8°$ and it has to rise 18 inches, how long must the ramp be?

9. A kite has 25 feet of string. The wind is blowing the kite to the west so that the angle of elevation is $40°$. How far has the kite traveled horizontally?

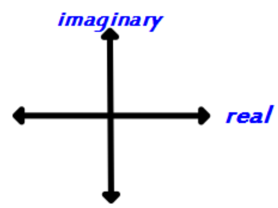
10. A sledding run is 400 yards long with a vertical drop of 40.2 yards. Find the angle of depression of the run.

Unit 5 Cluster 5 Honors (N.CN.3, N.CN.4, N.CN.5 and N.CN.6)
Using Complex Numbers in Rectangular and Polar Form

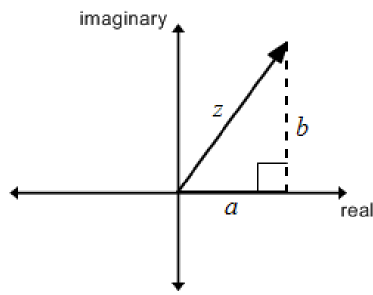
H.5.1 Find moduli of complex numbers.
H.5.2 Represent complex numbers on the complex plane in rectangular and polar form and explain why the rectangular and polar forms of a given complex number represent the same number.
H.5.3 Represent addition, subtraction, multiplication, and conjugation of complex number geometrically on the complex plane; use properties of this representation for computation.
H.5.4 Calculate the distance between numbers in the complex plane as the modulus of the difference
H.5.4 Calculate the midpoint of a segment as the average of the numbers at its endpoints.

VOCABULARY

The **complex plane** is where the horizontal axis represents the real component, *a*, and the vertical axis represents the imaginary component, *bi*, of a complex number.



The **rectangular form** of a complex number, $a+bi$, is written as (a,b). Traditionally known as (x,y). To graph a point in the complex plane, graph *a* units horizontally and *b* units vertically.

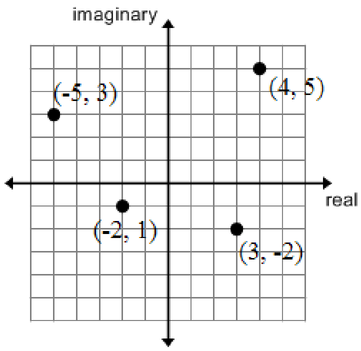


The **modulus** (plural form is moduli) of a complex number is defined by $|z|=\sqrt{a^2+b^2}$ and is the length, or magnitude, of the vector in component form created by the complex number in the complex plane.

<u>Example 1:</u>
Write the following complex numbers in rectangular form, then graph them on the complex plane.

a. $3-2i$
b. $4+5i$
c. $-2-i$
d. $-5+3i$

Answer:

a. $3-2i \rightarrow (3,-2)$ b. $4+5i \rightarrow (4,5)$ c. $-2-i \rightarrow (-2,-1)$ d. $-5+3i \rightarrow (-5,3)$	The rectangular form of complex number $a+bi$ is written as (a,b).
	Since *a* is the real component it is graphed horizontally. Similarly, *bi* is the complex component and is graphed vertically. a. 3 units right and 2 units down b. 4 units right and 5 units up c. 2 units left and 1 unit down d. 5 units left and 3 units up

Example 2:
Find the modulus of $z=3-4i$.

$z=3-4i$	$a=3$ and $b=-4$
$\|z\|=\sqrt{3^2+(-4)^2}$	Modulus formula: $\|z\|=\sqrt{a^2+b^2}$
$\|z\|=\sqrt{9+16}$ $\|z\|=\sqrt{25}$ $\|z\|=5$	Simplify.

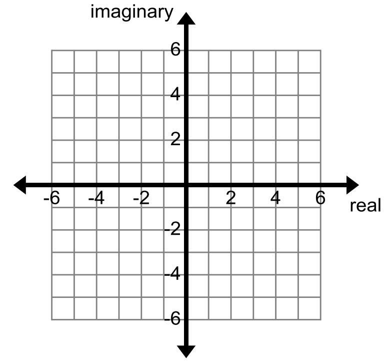
Example 3:
Find the modulus of the complex number in rectangular form $(3,-6)$.

$(3,-6)$	$a=3$ and $b=-6$
$\|z\|=\sqrt{3^2+(-6)^2}$	Modulus formula: $\|z\|=\sqrt{a^2+b^2}$
$\|z\|=\sqrt{9+36}$ $\|z\|=\sqrt{45}\approx 6.71$	Simplify.

Practice Exercises A

Write the following complex numbers in rectangular form then graph them on the complex plane.

1. $-2+3i$
2. $5+4i$
3. $-1-2i$
4. $3-5i$
5. $1-i$
6. $-5-6i$

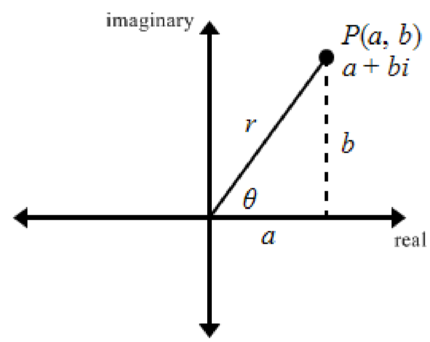


Find the moduli of the following complex numbers.

7. $-5+12i$
8. $5+4i$
9. $(-1,-2)$
10. $(8,-6)$

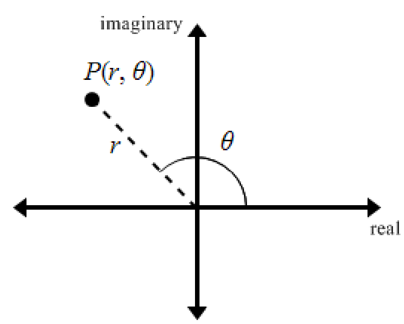
VOCABULARY

The **polar form** of a complex number, $z=a+bi$, is represented by $z=r(\cos\theta+i\sin\theta)$ (sometimes called trigonometric form) where $a=r\cos\theta$, $b=r\sin\theta$, r is the modulus, $r=|z|=\sqrt{a^2+b^2}$, of a complex number and θ is an argument of *z*. An **argument** of a complex number, $z=a+bi$, is the direction angle of the vector $\langle a,b\rangle$.



To find θ, when given a complex number $z=a+bi$, use $\theta=\tan^{-1}\left(\frac{b}{a}\right)$.

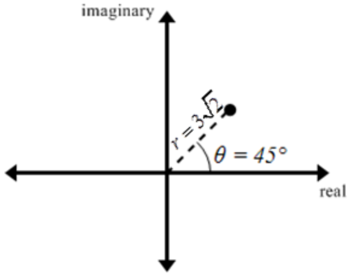
To graph a complex number in polar form, plot a point *r* units from the origin on the positive *x*-axis, then rotate the point the measure of θ.



To convert from polar to rectangular form use the fact that $a=r\cos\theta$ and $b=r\sin\theta$.

Example 4:
Write $3+3i$ in polar form then graph it.

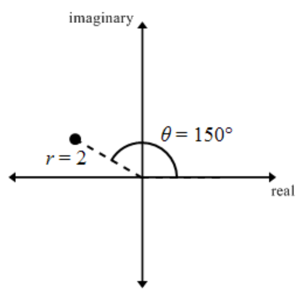
Answer

$3+3i$	$a=3$ and $b=3$
$\lvert z\rvert=\sqrt{3^2+3^2}$ $\lvert z\rvert=\sqrt{9+9}$ $\lvert z\rvert=\sqrt{18}$ $\lvert z\rvert=3\sqrt{2}$	Compute the modulus $\lvert z\rvert=\sqrt{a^2+b^2}$.
$\theta=\tan^{-1}\left(\frac{3}{3}\right)$ $\theta=45°$	Find θ. Make sure that your calculator is in degrees.
$3\sqrt{2}\left(\cos 45°+i\sin 45°\right)$	Remember that a complex number in polar form is $z=r\left(\cos\theta+i\sin\theta\right)$. The modulus is r so $r=3\sqrt{2}$.
	Plot the point $3\sqrt{2}$ units from the origin on the positive *x*-axis. Rotate it $45°$.

Example 5:
Write $\left(-\sqrt{3},1\right)$ in polar form then graph it.

Answer:

$-\sqrt{3}+i$	$a=-\sqrt{3}$ and $b=1$
$\lvert z\rvert=\sqrt{\left(-\sqrt{3}\right)^2+1^2}$ $\lvert z\rvert=\sqrt{3+1}$ $\lvert z\rvert=\sqrt{4}$ $\lvert z\rvert=2$	Compute the modulus $\lvert z\rvert=\sqrt{a^2+b^2}$.
$\theta=\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$ $\theta=-30°$	Find θ. Make sure that your calculator is in degrees.

$\theta = -30° + 180° = 150°$	The calculator only returns values between $-90°$ and $90°$. In order to get the correct angle, so that the point is in quadrant II, add $180°$ to the angle. (If the point is in quadrant III, you will also add $180°$, but if it is in quadrant IV, you will need to add $360°$.)
$z = 2(\cos 150° + i \sin 150°)$	Remember that a complex number in polar form is $z = r(\cos\theta + i\sin\theta)$. The modulus is r so $r = 2$.
	Plot the point 2 units from the origin on the positive *x*-axis. Rotate the point from $150°$ the positive *x*-axis.

Example 6:
Write $\sqrt{2}(\cos 135° + i \sin 135°)$ in rectangular form.

Answer:

$\sqrt{2}(\cos 135° + i \sin 135°)$	$r = \sqrt{2}$ and $\theta = 135°$
$a = \sqrt{2}\cos 135° = -1$ $b = \sqrt{2}\sin 135° = 1$	To convert from polar to rectangular form use $a = r\cos\theta$ and $b = r\sin\theta$. Make sure that your calculator is in degree mode.
The rectangular form is $(-1,1)$.	Rectangular form is (a,b).

Example 7:
Write $\sqrt{3}(\cos 210° + i \sin 210°)$ in rectangular form.

Answer:

$\sqrt{3}(\cos 210° + i \sin 210°)$	$r = \sqrt{3}$ and $\theta = 210°$
$a = \sqrt{3}\cos 210° = \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3}{2} - 1.5$ $b = \sqrt{3}\sin 210° = \sqrt{3}\left(-\frac{1}{2}\right) = -\frac{\sqrt{3}}{2} \approx -0.866$	To convert from polar to rectangular form use $a = r\cos\theta$ and $b = r\sin\theta$. Make sure that your calculator is in degree mode.
The rectangular form is $\left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$.	Rectangular form is (x, y).

Practice Exercises B

Write the following complex numbers in polar form.

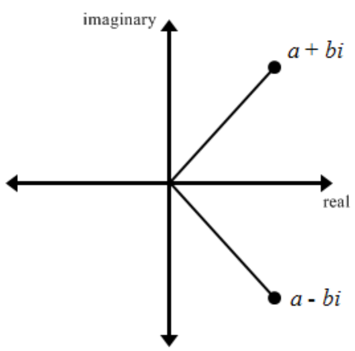
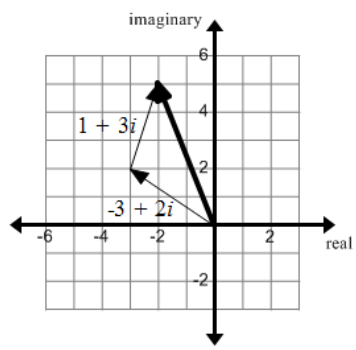
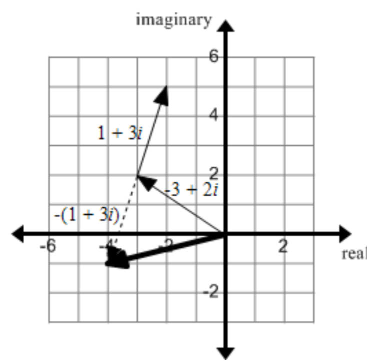
1. $-5+12i$ 2. $5+4i$ 3. $(0,2)$ 4. $(8,-6)$

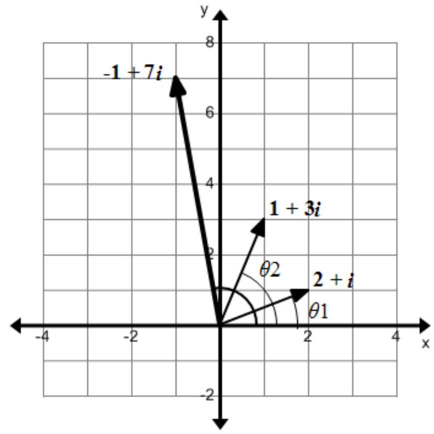
Write the following complex numbers in rectangular form.

5. $3(\cos 270° + i\sin 270°)$
6. $\sqrt{6}(\cos 60° + i\sin 60°)$
7. $3(\cos 135° + i\sin 135°)$
8. $4(\cos 30° + i\sin 30°)$

9. a. Write $10+24i$ in rectangular form.
   b. Write your answer to part a) in polar form.
   c. Write your answer to part b) in rectangular form.
   d. Compare your answers form part a) and part c).

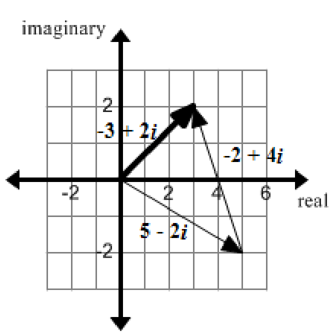
Representing Addition, Subtraction and Multiplication of Complex Numbers

A Complex Number and its Conjugate	**Adding Complex Numbers**	**Subtracting a Complex Number**
 The conjugate is a reflection through the x-axis.	 Adding complex numbers can be demonstrated geometrically by showing the addition of the vector associated with each complex number z. The sum of $(-3+2i)+(1+3i)$ is the resultant vector $\langle -2,5\rangle$ or $-2+5i$.	 Subtracting complex numbers can be demonstrated geometrically by showing the subtraction of the vector associated with each complex number z. The difference of $(-3+2i)-(1+3i)$ is the resultant vector $\langle -4,-1\rangle$ or $-4-i$.

Multiplying a Complex Number	
Multiplication of complex numbers is geometrically represented by adding the angles and multiplying the moduli of the two vectors. $(2+i)(1+3i)=2+7i-3=-1+7i$ $\theta=\tan^{-1}\left(\frac{7}{-1}\right)\approx -81.870°+180°=98.13°$ $\lvert z\rvert=\sqrt{(-1)^2+7^2}=\sqrt{1+49}=\sqrt{50}\approx 7.071$	
$z_1=(2+i)$ $\theta_1=\tan^{-1}\left(\frac{1}{2}\right)\approx 26.6°$ $\lvert z_1\rvert=\sqrt{2^2+1^2}=\sqrt{4+1}=\sqrt{5}$	$z_2=(1+3i)$ $\theta_2=\tan^{-1}\left(\frac{3}{1}\right)\approx 71.6°$ $\lvert z_2\rvert=\sqrt{1^2+3^2}=\sqrt{1+9}=\sqrt{10}$
Sum of angles $\theta=\theta_1+\theta_2=26.565°+71.565°=98.13°$ Product of moduli: $\lvert z\rvert=\lvert z_1\rvert\cdot\lvert z_2\rvert=\sqrt{5}\cdot\sqrt{10}=\sqrt{50}\approx 7.071$	

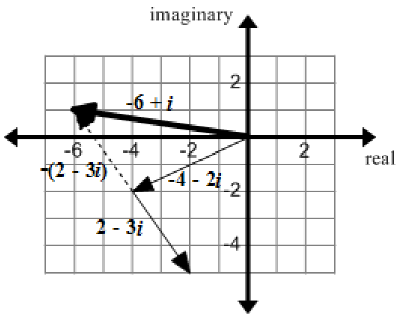
<u>Example 8:</u>
Add $(5-2i)+(-2+4i)$ and represent the sum by graphing it on the complex plane.

Answer:

$(5-2i)+(-2+4i)$	
$5+(-2i)+(-2)+4i$	Remove the parenthesis.
$5+(-2)+(-2i)+4i$ $3+2i$	Combine like terms.
	Represent the two complex numbers with vectors.

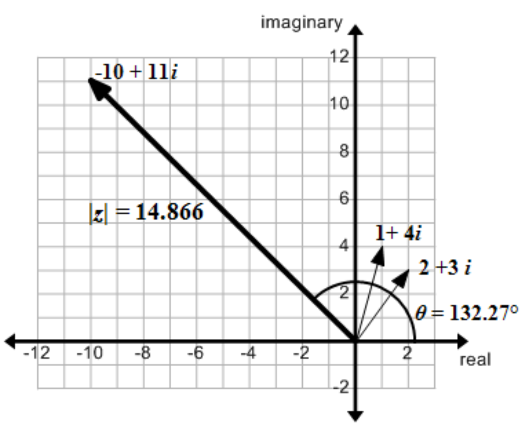
Example 9:
Subtract $(-4-2i)-(2-3i)$ and represent the difference by graphing it on the complex plane.

Answer:

$(-4-2i)-(2-3i)$	
$-4+(-2i)+(-2)+3i$	Remove the parenthesis.
$-4+(-2)+(-2i)+3i$ $-6+i$	Combine like terms.
	Represent the two complex numbers with vectors.

Example 10:
Multiply $(2+3i)(1+4i)$ and represent the product by graphing it on the complex plane.

Answer:

$(2+3i)(1+4i)$	
$2+3i+8i+12i^2$ $2+11i-12$ $-10+11i$	Use your preferred method to multiply. Simplify and remember that $i^2=-1$.
	$\theta_1=\tan^{-1}\left(\frac{3}{2}\right)=56.310°$ $\theta_2=\tan^{-1}\left(\frac{4}{1}\right)=75.964°$ Sum of angles: $56.310°+75.964°=132.27°$ $\lvert z_1\rvert=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}$ $\lvert z_2\rvert=\sqrt{1^2+4^2}=\sqrt{1+16}=\sqrt{17}$ Product of moduli: $\sqrt{13}\cdot\sqrt{17}=\sqrt{221}\approx 14.866$

Practice Exercises C

Find the conjugate of the complex numbers.

1. $-9i$
2. $-4+i$
3. $9-6i$
4. $-3-7i$

Add or subtract and represent the sum or difference on the complex plane.

5. $(8+5i)+(7+6i)$
6. $(-4+7i)+(-4i)$
7. $(3+6i)+(-4+6i)$
8. $(-1-3i)-(-6-4i)$
9. $(1-7i)-(5+8i)$
10. $(-5+2i)-(9+10i)$

Multiply and represent the product on the complex plane.

11. $(1+i)(2+i)$
12. $(3+2i)(4+3i)$
13. $(3+i)(5+2i)$

Powers of Complex Numbers

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and *n* is any positive integer, then $z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$.

Example 11:

Use De Moivre's Theorem to simplify $\left(-\sqrt{3}+i\right)^8$

Answer:

$\left(-\sqrt{3}+i\right)^8$	You need the radius and the angle. To get the radius find the modulus.
$\lvert z\rvert = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2}$ $\lvert z\rvert = \sqrt{3+1}$ $\lvert z\rvert = \sqrt{4}$ $\lvert z\rvert = 2$	$a = -\sqrt{3}$, $b = 1$, and $r = \lvert z\rvert = \sqrt{a^2+b^2}$.
$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$ $\theta = -30°$	To get the angle use $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.
$\theta = -30° + 180° = 150°$	The point should be in the second quadrant so add $180°$ to the angle.

$\left(-\sqrt{3}+i\right)^8 = 2^8\left(\cos\left(8\cdot 150°\right)+i\sin\left(8\cdot 150°\right)\right)$ $\left(-\sqrt{3}+i\right)^8 = 256\left(\cos\left(1200°\right)+i\sin\left(1200°\right)\right)$ $\left(-\sqrt{3}+i\right)^8 = 256\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ $\left(-\sqrt{3}+i\right)^8 = \left(-128+128\sqrt{3}i\right)$ $\left(-\sqrt{3}+i\right)^8 = -128+128\sqrt{3}i$	Substitute $r=2$, $\theta=150°$, and $n=8$ into De Moivre's Theorem.

Example 12:

Use De Moivre's Theorem to simplify $\left(-1+\sqrt{3}i\right)^3$.

Answer:

$\left(-1+\sqrt{3}i\right)^3$	You need the radius and the angle. To get the radius find the modulus.
$\lvert z\rvert = \sqrt{(-1)^2+\left(\sqrt{3}\right)^2}$ $\lvert z\rvert = \sqrt{1+3}$ $\lvert z\rvert = \sqrt{4}$ $\lvert z\rvert = 2$	$a=-1,\ b=\sqrt{3},$ and $r=\lvert z\rvert=\sqrt{a^2+b^2}$.
$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ $\theta = -60°$	To get the angle use $\theta=\tan^{-1}\left(\frac{b}{a}\right)$.
$\theta = -60°+180° = 120°$	The point should be in the second quadrant so add $180°$ to the angle.
$\left(-1+\sqrt{3}i\right)^3 = 2^3\left(\cos\left(3\cdot 120°\right)+i\sin\left(3\cdot 120°\right)\right)$ $\left(-1+\sqrt{3}i\right)^3 = 8\left(\cos\left(360°\right)+i\sin\left(360°\right)\right)$ $\left(-1+\sqrt{3}i\right)^3 = 8\left(1+0i\right)$ $\left(-1+\sqrt{3}i\right)^3 = \left(8+0\right)$ $\left(-1+\sqrt{3}i\right)^3 = 8$	Substitute $r=2$, $\theta=120°$, and $n=3$ into De Moivre's Theorem.

Practice Exercises D

Use De Moivre's Theorem to simplify the following.

1. $(2+2i)^3$
2. $(1-i)^3$
3. $(2+3i)^4$
4. $(2-i)^4$
5. $(\sqrt{3}-i)^4$
6. $(-3-3i\sqrt{3})^5$

VOCABULARY

The **distance** between numbers in the complex plane is the modulus, $|z|=\sqrt{a^2+b^2}$, of the difference of the complex numbers.

The **midpoint** of a segment in the complex plane is the average of the endpoints.

Example 13:
Find the distance between $(-6-2i)$ and $(2-3i)$.
Answer:

$(-6-2i)-(2-3i)$ $-6-2i-2+3i$ $-8+i$	Find the difference between the two points.
$\lvert z\rvert=\sqrt{(-8)^2+(1)^2}$ $\lvert z\rvert=\sqrt{64+1}$ $\lvert z\rvert=\sqrt{65}\approx 8.06$	Compute the modulus, $\lvert z\rvert=\sqrt{a^2+b^2}$, of the difference. $a=-8$ and $b=1$

Example 14:
Find the midpoint between $(8-2i)$ and $(1+4i)$.
Answer:

$(8-2i)$ and $(1+4i)$	The endpoints of the first complex number are $a_1=8$ and $b_1=-2$. The endpoints of the second complex number are $a_2=1$ and $b_2=4$.
$\left(\frac{8+1}{2},\frac{-2+4}{2}\right)$ $\left(\frac{9}{2},\frac{2}{2}\right)$ $\left(\frac{9}{2},1\right)$ is the midpoint or $\left(\frac{9}{2}+i\right)$	Compute the midpoint by averaging the *a* endpoints and the *b* endpoints. $\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)$

Practice Exercises E

Find the distance and midpoint between the complex numbers.

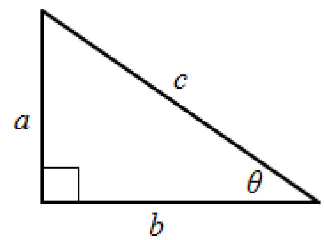
1. $(-3+2i)$ and $(1+3i)$
2. $(7-4i)$ and $(-2-2i)$
3. $(5-12i)$ and $(-10+7i)$
4. $(-3-8i)$ and $(-2+12i)$
5. $(-2-7i)$ and $(-7-9i)$
6. $(11-3i)$ and $(-3+3i)$

Unit 5 Cluster 6 (F.TF.8)
Using the Pyathagorean Identity

Cluster 6: Prove and apply trigionometric identities

5.6.1 Prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta)=1$, then use it to find sin(θ), cos(θ), or tan(θ), given sin(θ), cos(θ), or tan(θ) (limit angles between 0 and 90 degrees).

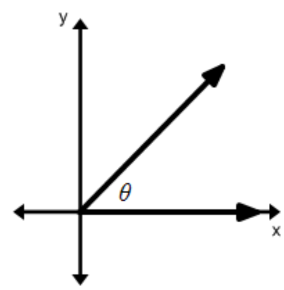
Proof of the Pythagorean Identity $\sin^2(\theta)+\cos^2(\theta)=1$



$a^2+b^2=c^2$	Given a right triangle with side lengths *a*, *b*, and *c*, use the Pythagorean Theorem to relate the sides of the triangle.
$\frac{a^2}{c^2}+\frac{b^2}{c^2}=\frac{c^2}{c^2}$ $\frac{a^2}{c^2}+\frac{b^2}{c^2}=1$	Divide each side by c^2 so that the expression on the left is equal to 1 on the right.
$\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$	Use the properties of exponents to rewrite the expression on the left.
$[\sin(\theta)]^2+[\cos(\theta)]^2=1$ $\sin^2(\theta)+\cos^2(\theta)=1$	Substitute $\sin(\theta)=\frac{a}{c}$ and $\cos(\theta)=\frac{b}{c}$.

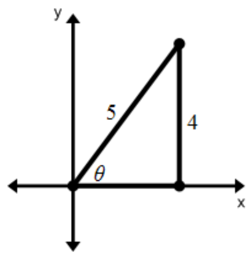
VOCABULARY

An angle is in **standard position** when the vertex is at the origin, one ray is on the positive *x*-axis, and the other ray extends into the first quadrant.



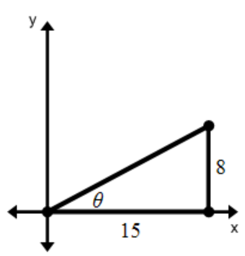
Example 1:

If $\sin(\theta) = \frac{4}{5}$ is in the first quadrant, find $\cos(\theta)$ and $\tan(\theta)$.

	Draw a triangle with the angle θ in standard position. Then label the information you know.
$4^2 + b^2 = 5^2$ $16 + b^2 = 25$ $b^2 = 25 - 16$ $b^2 = 9$ $b = 3$	Use the Pythagorean Theorem to find the missing side.
$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$ $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$	Use the definitions of the trigonometric ratios to find $\cos(\theta)$ and $\tan(\theta)$.

Example 2:

If $\tan(\theta) = \frac{8}{15}$ is in the first quadrant, find $\sin(\theta)$ and $\cos(\theta)$.

	Draw a triangle with the angle θ in standard position. Then label the information you know.
$8^2 + 15^2 = c^2$ $64 + 225 = c^2$ $289 = c^2$ $c = 17$	Use the Pythagorean Theorem to find the missing side.
$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{8}{17}$ $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$	Use the definitions of the trigonometric ratios to find $\sin(\theta)$ and $\cos(\theta)$.

Practice Exercises A

1. Find $\sin(\theta)$ and $\cos(\theta)$ if $\tan(\theta)=\frac{3}{4}$.

2. Find $\sin(\theta)$ and $\cos(\theta)$ if $\tan(\theta)=3$.

3. Find $\sin(\theta)$ and $\tan(\theta)$ if $\cos(\theta)=\frac{1}{4}$.

4. Find $\sin(\theta)$ and $\tan(\theta)$ if $\cos(\theta)=\frac{4}{5}$.

5. Find $\cos(\theta)$ and $\tan(\theta)$ if $\sin(\theta)=\frac{5}{13}$.

6. Find $\cos(\theta)$ and $\tan(\theta)$ if $\sin(\theta)=\frac{1}{\sqrt{2}}$.

7. Find $\sin(\theta)$ and $\cos(\theta)$ if $\tan(\theta)=\frac{8}{5}$.

8. Find $\sin(\theta)$ and $\tan(\theta)$ if $\cos(\theta)=\frac{1}{2}$.

9. Find $\cos(\theta)$ and $\tan(\theta)$ if $\sin(\theta)=\frac{\sqrt{3}}{2}$

10. Find $\sin(\theta)$ and $\cos(\theta)$ if $\tan(\theta)=\sqrt{3}$.

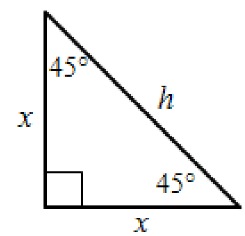
Unit 5 Honors Unit Circle
Defining Trigonometric Ratios on the Unit Circle

H.5.6 Define trigonometric ratios and write trigonometric expressions in equivalent forms.

There are special right triangles. These triangles have special relationships between the lengths of their sides and their angles that can be used to simplify calculations when finding missing angles and sides.

$45° - 45° - 90°$ Triangle

The Pythagorean Theorem allows us to derive the relationships that exist for these triangles. Consider a right isosceles triangle with leg lengths *x* and hypotenuse *h*. Since this is a right isosceles triangle, the measures of the angles are $45° - 45° - 90°$.

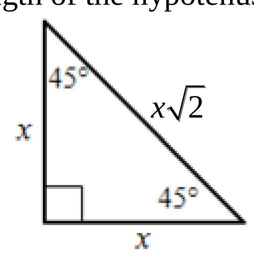


Using the Pythagorean Theorem, we know that $x^2 + x^2 = h^2$. Solving the equation for *h*, we get:

$$h^2 = x^2 + x^2$$
$$h^2 = 2x^2$$
$$h = \sqrt{2x^2}$$
$$h = x\sqrt{2}$$

$45° - 45° - 90°$ Triangle

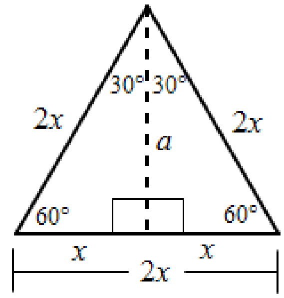
In any $45° - 45° - 90°$ triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of its leg.



Therefore, in a $45° - 45° - 90°$, the measures of the side lengths are *x*, *x*, and $x\sqrt{2}$.

$30^\circ - 60^\circ - 90^\circ$ Triangles

There is also a special relationship for triangles with angles of $30^\circ - 60^\circ - 90^\circ$. When an altitude, a, is drawn from the vertex of an equilateral triangle, it bisects the base of the triangle and two congruent $30^\circ - 60^\circ - 90^\circ$ triangles are formed. If each triangle has a base of length x, then the entire length of the base of the equilateral triangle is $2x$.



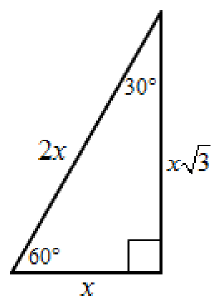
Using one of the right triangles and the Pythagorean Theorem, we know that $a^2 + x^2 = (2x)^2$. Solving for a we get:

$$a^2 + x^2 = (2x)^2$$
$$a^2 = 4x^2 - x^2$$
$$a^2 = 3x^2$$
$$a = \sqrt{3x^2}$$
$$a = x\sqrt{3}$$

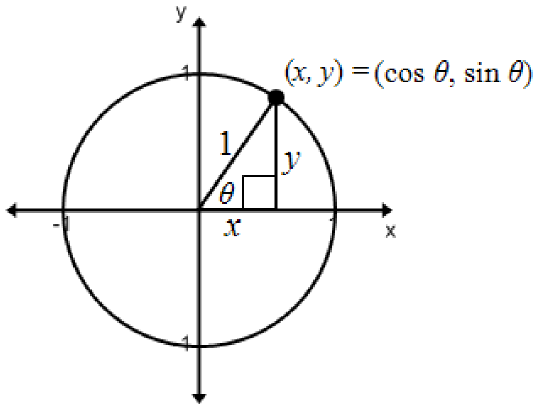
Therefore, in a $30^\circ - 60^\circ - 90^\circ$ triangle, the measures of the side lengths are x, $x\sqrt{3}$, and $2x$.

$30^\circ - 60^\circ - 90^\circ$ Triangles

In any $30^\circ - 60^\circ - 90^\circ$ triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.



Trigonometric Ratios for a Circle of Radius 1



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sin\theta = \frac{y}{1} \qquad \cos\theta = \frac{x}{1}$$

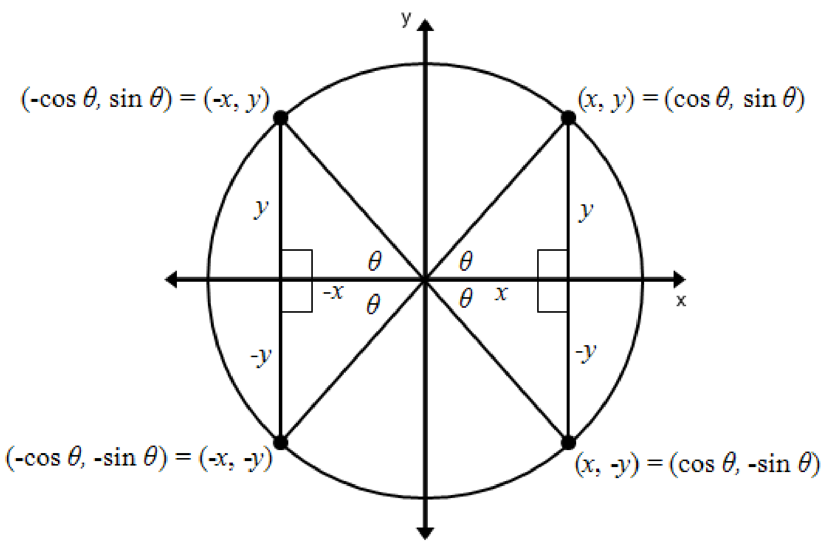
$$\sin\theta = y \qquad \cos\theta = x$$

For any point (x, y) on a circle of radius 1, the *x*-coordinate is the cosine of the angle and the *y*-coordinate is the sine of the angle. A circle of radius 1 is called a unit circle. Special values on the unit circle are derived from the special right triangles, $45°-45°-90°$ triangles and $30°-60°-90°$ triangles, and their trigonometric ratios.

Special Right Triangles and the Unit Circle

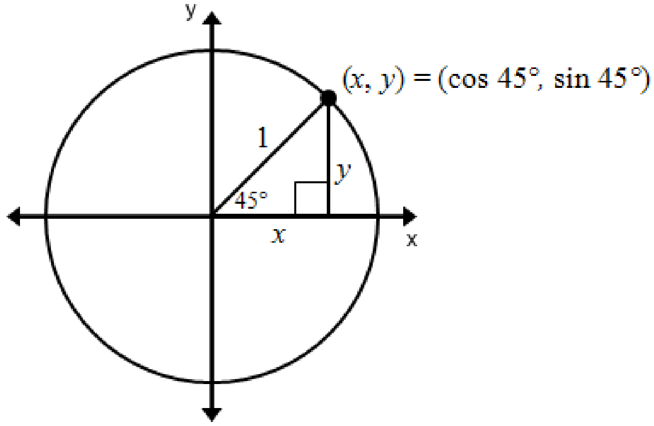
Recall that if you reflect any point (x, y), on the coordinate plane over the *x* axis, *y* axis, or the origin, then the following relationships exist:

Reflection over the *y* axis $(x, y) \to (-x, y)$
Reflection over the *x* axis $(x, y) \to (x, -y)$
Reflection over the origin $(x, y) \to (-x, -y)$



We will use this reasoning to show relationships of special right triangles on the unit circle.

To illustrate a $45°-45°-90°$ triangle on the unit circle, we are going to rotate a point $45°$ from the positive *x*-axis. The right triangle formed has a hypotenuse of 1 and leg lengths of *x* and *y*.



Since two of the angles are congruent this is an isosceles right triangle and the lengths of the legs are the same, $x = y$. Finding the value of *x* will help us identify the numerical coordinates of the point (x, y).

The length of the hypotenuse in a $45°-45°-90°$ is equal to the length of a leg times $\sqrt{2}$. Using this property we can find the length of a leg.

$1 = x\sqrt{2}$	The hypotenuse, which is length 1, is $\sqrt{2}$ times the length of leg *x*.
$\frac{1}{\sqrt{2}} = x$	Solve for *x*.
$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = x$	
$\frac{\sqrt{2}}{2} = x$	Rationalize the denominator.

Since both legs are equal, $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$.

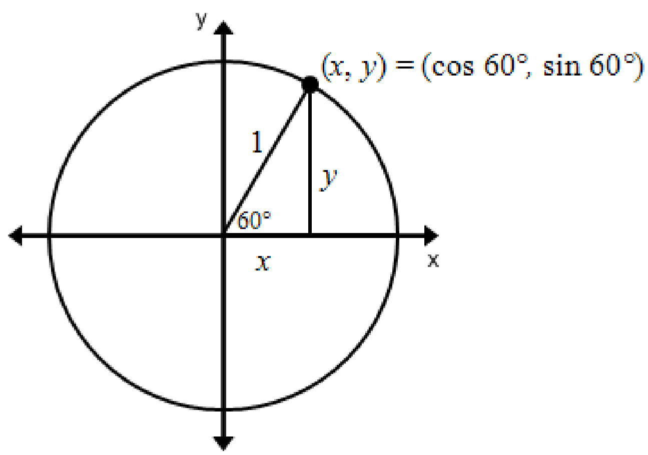
If we relate this to the unit circle where $\theta = 45°$, then the following is true:

$$x = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \cos 45° = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2} \qquad y = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \sin 45° = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

Thus, the point $(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point over the *y*-axis is equivalent to a rotation of $135°$ from the positive *x*-axis. The coordinates of the point are
  $(-x, y) = (-\cos 45°, \sin 45°) = (\cos 135°, \sin 135°) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $225°$ from the positive *x*-axis. The coordinates of the new point are
  $(-x, -y) = (-\cos 45°, -\sin 45°) = (\cos 225°, \sin 225°) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.
- The reflection of the point over the *x*-axis is equivalent to a rotation of $315°$ from the positive *x*-axis. The coordinates of the new point are
  $(x, -y) = (\cos 45°, -\sin 45°) = (\cos 315°, \sin 315°) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

To illustrate a $30°-60°-90°$ triangle on the unit circle, we are going to rotate a point $60°$ from the positive *x*-axis and drop a perpendicular line to the positive *x*-axis. The right triangle formed has a hypotenuse of 1 and leg lengths of *x* and *y*.



The length of the hypotenuse in a $30°-60°-90°$ is twice the length of the shorter side. Thus, $1 = 2x$. Upon solving the equation for *x* we find that $x = \frac{1}{2}$. We can use this value of *x* to find the length of the longer leg, *y*. The longer leg is $\sqrt{3}$ times the length of the shorter leg. Therefore, $y = x\sqrt{3}$ and since $x = \frac{1}{2}$ then $y = \left(\frac{1}{2}\right)\sqrt{3}$ or $y = \frac{\sqrt{3}}{2}$.

If we relate this to the unit circle where $\theta = °60$ then the following is true:

$$x = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \cos 60° = \frac{\frac{1}{2}}{1} = \frac{1}{2} \qquad y = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \sin 60° = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

Thus, the point $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point over the *y*-axis is equivalent to a rotation of $120°$ from the positive *x*-axis. The coordinates of the point are $(-x, y) = \left(-\cos 60°, \sin 60°\right) = \left(\cos 120°, \sin 120°\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $240°$ from the positive *x*-axis. The coordinates of the new point are $(-x, -y) = \left(-\cos 60°, -\sin 60°\right) = \left(\cos 240°, \sin 240°\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.
- The reflection of the point over the *x*-axis is equivalent to a rotation of $300°$ from the positive *x*-axis. The coordinates of the new point are $(x, -y) = \left(\cos 60°, -\sin 60°\right) = \left(\cos 300°, \sin 300°\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

We can also consider the case where $\theta = °30$ and use the $30° - 60° - 90°$ triangle to find the values of *x* and *y* for this value of θ.

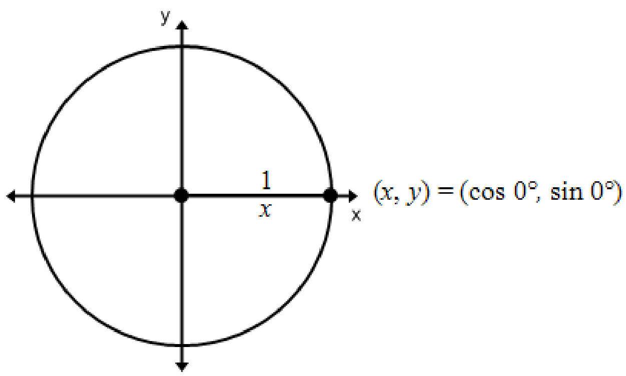
$$x = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \cos 30° = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \qquad y = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \sin 30° = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

Thus, the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point over the *y*-axis is equivalent to a rotation of $150°$ from the positive *x*-axis. The coordinates of the point are $(-x, y) = \left(-\cos 30°, \sin 30°\right) = \left(\cos 150°, \sin 150°\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

- The reflection of the point through the origin is equivalent to a rotation of $210°$ from the positive x-axis. The coordinates of the new point are $(-x,-y)=(-\cos 30°,-\sin 30°)=(\cos 210°,\sin 210°)=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$.
- The reflection of the point over the x-axis is equivalent to a rotation of $330°$ from the positive x-axis. The coordinates of the new point are $(x,-y)=(\cos 30°,-\sin 30°)=(\cos 330°,\sin 330°)=\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$.

If we plot all of the points where $\theta = 30°,\ 45°,$ and $60°$ and their reflections, then we get most of the unit circle. To obtain the rest of the unit circle we have to examine what happens to a point when $\theta = 0°$.

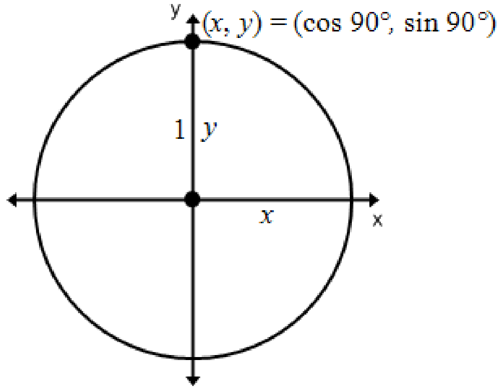


Notice that the value of x is equal to 1 and that the value of y is zero. Thus, when $\theta = 0°$, the point $(x, y)=(1,0)$. Even though this point does not form a right triangle, any point on a circle can be found by using cosine and sine. Therefore, $\cos 0° = 1$ and $\sin 0° = 0$.

If we reflect the point over the y-axis, then the new point is $(-x, y)=(-1,0)$. This is equivalent to a rotation of $180°$ from the positive x-axis. The coordinates of the new point are:

$$(-x, y)=(-\cos 0°,\sin 0°)=(\cos 180°,\sin 180°)=(-1,0).$$

Finally we need to observe what happens when we rotate a point $90°$ from the positive *x*-axis.

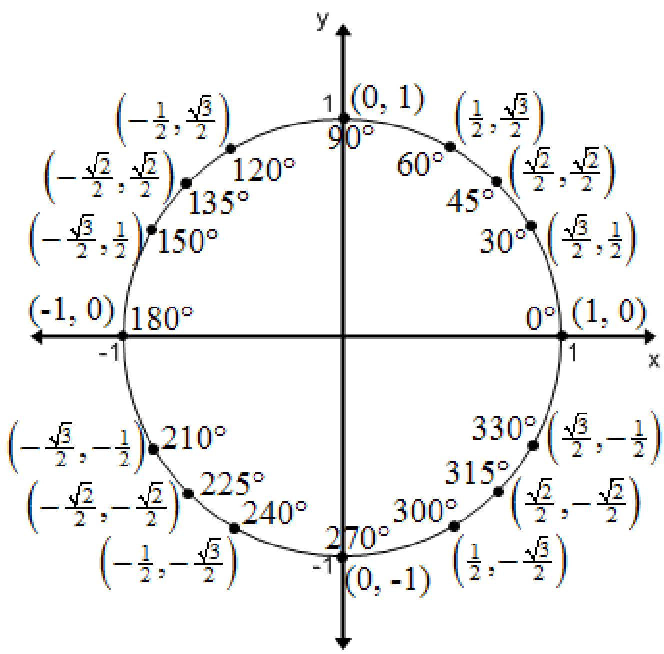


Notice that the value of *y* is equal to 1 and that the value of *x* is zero. Thus, when $\theta = 90°$, the point $(x, y) = (0,1)$. Therefore, $\cos 90° = 0$ and $\sin 90° = 1$.

If we reflect the point over the *x*-axis, then the new point is $(x, -y) = (0, -1)$. This is equivalent to a rotation of $270°$ from the positive *x*-axis. The coordinates of the new point are:

$$(x, -y) = (\cos 90°, \sin 90°) = (\cos 270°, \sin 270°) = (0, -1).$$

Plotting all of the points, we obtain what is referred to as the unit circle.



The unit circle can be used to find exact values of trigonometric ratios for the angles that relate to the special right triangle angles.

Example 1:
Find $\sin 135°$.

Answer:

The point that has been rotated $135°$ from the positive *x*-axis has coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The *y*-coordinate is the sine value, therefore, $\sin 135° = \frac{\sqrt{2}}{2}$.

Example 2:
Find $\cos 240°$.

Answer:

The point that has been rotated $240°$ from the positive *x*-axis has coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The *x*-coordinate is the cosine value, therefore, $\cos 240° = -\frac{1}{2}$.

Example 3:

Find all values of θ, $0° \le \theta \le 360°$, for which $\sin\theta = \frac{1}{2}$.

Answer:
The points that have been rotated $30°$ and $150°$ from the positive *x*-axis have coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ respectively. The y-coordinate is the sine value and both points have a *y*-coordinate of $\frac{1}{2}$.

Example 4:
Find all values of θ, $0° \le \theta \le 360°$, for which $\cos\theta = -1$.

Answer:
The point that has been rotated $180°$ from the positive *x*-axis has coordinates $(-1, 0)$. The *x*-coordinate is the cosine value which is -1.

Defining Tangent Values

Another way to write $\tan\theta$ is $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$. This can be shown algebraically as follows:

$\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}}$	Use the definition $\sin\theta=\dfrac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta\dfrac{\text{adjacent}}{\text{hypotenuse}}$.
$\tan\theta=\dfrac{\text{opposite}}{\text{hypotenuse}}\div\dfrac{\text{adjacent}}{\text{hypotenuse}}$	Rewrite the division problem so that it is easier to work with.
$\tan\theta=\dfrac{\text{opposite}}{\text{hypotenuse}}\cdot\dfrac{\text{hypotenuse}}{\text{adjacent}}$	Dividing by a fraction is the same as multiplying by its reciprocal.
$\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}\cdot\dfrac{\text{hypotenuse}}{\text{hypotenuse}}$	Use the commutative property of multiplication to rearrange the terms.
$\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}\cdot 1$	$\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}=\dfrac{\sin\theta}{\cos\theta}$

<u>Example 5:</u>

Find $\tan 210°$.

Answer:

The coordinates of the point that has been rotated $210°$ from the positive *x*-axis are $\left(-\dfrac{\sqrt{3}}{2},-\dfrac{1}{2}\right)$.

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

$$\tan 210°=\frac{\sin 210°}{\cos 210°}$$

Use the coordinates of the point to find $\tan 210°$.

$$\tan 210°=\frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$

$\sin 210°=-\dfrac{1}{2}$ and $\sin 210°=-\dfrac{\sqrt{3}}{2}$

$$\tan 210°=-\frac{1}{2}\div-\frac{\sqrt{3}}{2}$$

Rewrite the division problem.

$$\tan 210°=-\frac{1}{2}\cdot-\frac{2}{\sqrt{3}}$$

Dividing by a fraction is the same as multiplying by the reciprocal.

$$\tan 210°=-\frac{1}{\sqrt{3}}$$

Simplify.

$$\tan 210°=-\frac{\sqrt{3}}{3}$$

Rationalize the denominator.

Tangent Values for the Angles on the Unit Circle

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$
θ	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$

Previously we defined the six trigonometric functions. Notice that cosecant, secant, and cotangent are reciprocals of sine, cosine, and tangent, respectively.

The Six Trigonometric Functions

sine $(\theta) = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$

cosecant $(\theta) = \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$

cosine $(\theta) = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

secant $(\theta) = \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$

tangent $(\theta) = \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

cotangent $(\theta) = \cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}}$

Example 6:

Find $\sec 60°$.

Answer:

$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2}$ and secant is the reciprocal of cosine therefore,

$\sec 60° = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{1} = 2$.

Example 7:

Find $\cot 330°$.

Answer:

$\tan 330° = \frac{\sin 330°}{\cos 330°} = -\frac{\sqrt{3}}{3}$ and cotangent is the reciprocal of tangent therefore,

$\cot 330° = \frac{\cos 330°}{\sin 330°} = -\frac{3}{\sqrt{3}} = -\frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = -\sqrt{3}$.

Example 8:

Find all values of θ, $0° \le \theta \le 360°$, for which $\csc\theta = \frac{2}{\sqrt{3}}$.

Answer:

Cosecant is the reciprocal of sine, therefore find all the values that satisfy $\sin\theta = \frac{\sqrt{3}}{2}$. The angles rotated $60°$ and $120°$ from the positive *x*-axis have coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ respectively. Both have *y*-coordinates of $\frac{\sqrt{3}}{2}$. Both will have a cosecant of $\frac{2}{\sqrt{3}}$.

Practice Exercises A

Find the value indicated.

1.	$\sin 135°$	2.	$\cos 270°$	3.	$\tan 300°$	4.	$\sin 45°$
5.	$\cos 60°$	6.	$\tan 120°$	7.	$\sin 180°$	8.	$\cos 0°$
9.	$\tan 210°$	10.	$\sin 240°$	11.	$\cos 225°$	12.	$\tan 315°$
13.	$\csc 330°$	14.	$\sec 30°$	15.	$\cot 150°$	16.	$\csc 90°$
17.	$\sec 180°$	18.	$\cot 315°$	19.	$\csc 210°$	20.	$\sec 225°$
21.	$\cot 270°$	22.	$\csc 45°$	23.	$\sec 120°$	24.	$\cot 90°$

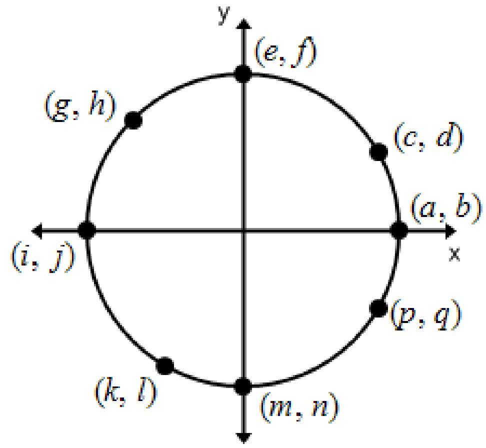
Find all values of θ, $0° \le \theta \le 360°$, that make the statement true.

25.	$\sin\theta = -\frac{\sqrt{2}}{2}$	26.	$\cos\theta = \frac{1}{2}$	27.	$\tan\theta = -1$	28.	$\sin\theta = -1$
29.	$\cos\theta = -\frac{\sqrt{3}}{2}$	30.	$\tan\theta = 0$	31.	$\csc\theta = -\frac{2}{\sqrt{3}}$	32.	$\sec\theta = 2$
33.	$\cot\theta = \sqrt{3}$	34.	$\csc\theta = 0$	35.	$\sec\theta = -1$	36.	$\cot\theta = 1$

Practice Exercises B

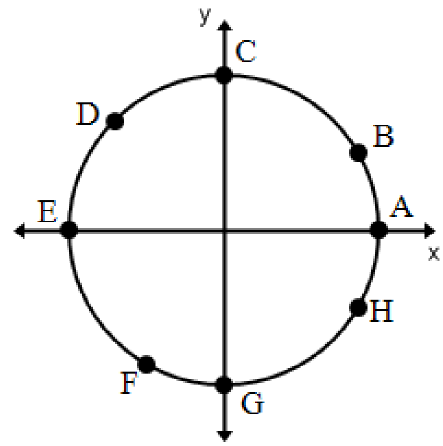
Refer to the diagram. Give the letter that could stand for the function value.

1. $\cos 180°$
2. $\sin 270°$
3. $\sin 30°$
4. $\cos 135°$
5. $\sin 0°$
6. $\cos 330°$
7. $\cos 90°$
8. $\sin 240°$
9. $\sin 135°$
10. $\cos 240°$
11. $\sin 330°$
12. $\cos 0°$



For the indicated point, tell if the value for $\sin\theta$ or $\cos\theta$ is positive, negative, or neither.

13. $\cos C$
14. $\sin G$
15. $\sin D$
16. $\cos H$
17. $\cos E$
18. $\sin B$
19. $\cos F$
20. $\cos B$
21. $\sin A$
22. $\cos G$
23. $\sin C$
24. $\sin E$



Unit 5 Honors Prove Trigonometric Identities
Trigonometry Proofs

H.5.7 Prove trigonometric identities using definitions, the Pythagorean Theorem, or other relationships.
H.5.7 Use the relationships to solve problems.

VOCABULARY

A **trigonometric identity** is a statement of equality that is true for all values of the variable for which both sides of the equation are defined. The set of values for which the variable is defined is called the **validity of the identity**.

The statement $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is an example of a trigonometric identity. The validity of the identity would not include values of θ that would make $\cos\theta = 0$ because dividing by zero is undefined.

Basic Trigonometric Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Cofunction Identities

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta \qquad \tan(90° - \theta) = \cot\theta$$

Negative Angle Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

Recall the work that you have done with expressions that are quadratic in nature. For example, $(x-3)^2 + 7(x-3) + 10$ is an expression that is quadratic in nature. If $u = x-3$, then the expression can be rewritten as $u^2 + 7u + 10$. This could then be factored as $(u+5)(u+2)$. Replacing u with $x-3$, you get $((x+3)+5)((x+3)+2)$. Trigonometry expressions can also be quadratic in nature. For example, the expression $\cos^2 x + 5\cos x + 6$ is quadratic in nature. If $u = \cos x$, then it could be rewritten as $u^2 + 5u + 6$. This could then be factored as $(u+2)(u+3)$. Substituting $\cos x$ back in for *u*, the factored expression is $(\cos x + 2)(\cos x + 3)$. You may need to use this idea when proving trigonometric identities.

Trigonometry Proofs

In a trigonometric proof you manipulate one side of the equation using the known trigonometric identities until it matches the other side of the equation. Pick the more complicated side to manipulate.

Example 1:
Prove the identity $1 = \sec^2 x - \tan^2 x$.

Answer:

$1 = \sec^2 x - \tan^2 x$	Manipulate the right side of the equation.
$1 = \dfrac{1}{\cos^2 x} - \dfrac{\sin^2 x}{\cos^2 x}$	Rewrite $\sec^2 x$ and $\tan^2 x$ using the reciprocal and quotient identities.
$1 = \dfrac{1 - \sin^2 x}{\cos^2 x}$	The fractions have a common denominator. Subtract the numerators.
$1 = \dfrac{\cos^2 x}{\cos^2 x}$	Us the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ to replace the numerator with $\cos^2\theta$.
$1 = 1$	Simplify.

Example 2:
Prove the trigonometric identity $\cos x - \cos x \sin^2 x = \cos^3 x$.

Answer:

$\cos x - \cos x \sin^2 x = \cos^3 x$	Manipulate the left side of the equation.
$\cos x\left(1 - \sin^2 x\right) = \cos^3 x$	Factor $\cos x$ from the two terms $\cos x$ and $\cos x \sin^2 x$.
$\cos x\left(\cos^2 x\right) = \cos^3 x$	Use the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$ to replace $1 - \sin^2 x$ with $\cos^2 x$.
$\cos^3 x = \cos^3 x$	Multiply.

Example 3:
Prove the identity $\sin x \tan x + \cos x = \sec x$.

Answer:

$\sin x \tan x + \cos x = \sec x$	Manipulate the left side of the equation.
$\sin x \cdot \frac{\sin x}{\cos x} + \cos x = \sec x$	Rewrite $\tan x$ using a quotient identity.
$\frac{\sin x}{1} \cdot \frac{\sin x}{\cos x} + \cos x = \sec x$ $\frac{\sin^2 x}{\cos x} + \cos x = \sec x$	Rewrite $\sin x$ so that it is a fraction. Multiply the fractions.
$\frac{\sin^2 x}{\cos x} + \frac{\cos x}{1} = \sec x$ $\frac{\sin^2 x}{\cos x} + \frac{\cos x}{1} \cdot \frac{\cos x}{\cos x} = \sec x$	Rewrite $\cos x$ so that it is a fraction. The common denominator is $\cos x \cdot 1 = \cos x$. Multiply the second fraction by $\frac{\cos x}{\cos x}$.
$\frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} = \sec x$	Simplify.
$\frac{\sin^2 x + \cos^2 x}{\cos x} = \sec x$	Add the numerators.
$\frac{1}{\cos x} = \sec x$	Use the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.
$\sec x = \sec x$	Use the reciprocal identities to rewrite the fraction.

Example 4:
Prove the identity $\tan x + \cot x = \sec x \csc x$.

Answer:

$\tan x + \cot x = \sec x \csc x$	Manipulate the left side of the equation.
$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \sec x \csc x$	Rewrite $\tan x$ and $\cot x$ using the quotient identities.
$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \sec x \csc x$ $\frac{\sin x}{\cos x} \cdot \frac{\sin x}{\sin x} + \frac{\cos x}{\sin x} \cdot \frac{\cos x}{\cos x} = \sec x \csc x$ $\frac{\sin^2 x}{\cos x \sin x} + \frac{\cos^2 x}{\cos x \sin x} = \sec x \csc x$	Find a common denominator in order to add the fractions. The common denominator is $\cos x \cdot \sin x = \cos x \sin x$. Multiply the first fraction by $\frac{\sin x}{\sin x}$ and the second fraction by $\frac{\cos x}{\cos x}$. Simplify.

$\dfrac{\sin^2 x+\cos^2 x}{\cos x\sin x}=\sec x\csc x$	Now that the fractions have a common denominator, add the numerators.
$\dfrac{1}{\cos x\sin x}=\sec x\csc x$	Use a Pythagorean Identity, $\cos^2\theta+\sin^2\theta=1$, to simplify the numerator.
$\dfrac{1}{\cos x}\cdot\dfrac{1}{\sin x}=\sec x\csc x$	Rewrite the single fraction as a product of two fractions.
$\sec x\cdot\csc x=\sec x\csc x$	Use the reciprocal identities to rewrite the fractions.

<u>Example 5:</u>

Prove the trigonometric identity $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$.

Answer:

$\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$	Manipulate the left side of the equation.
$\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$ $\dfrac{\cos x}{1+\sin x}\cdot\dfrac{\cos x}{\cos x}+\dfrac{1+\sin x}{\cos x}\cdot\dfrac{1+\sin x}{1+\sin x}=2\sec x$	Find a common denominator. The common denominator will be the product of the two denominators $(1+\sin x)\cos x$. Multiply the first fraction by $\dfrac{\cos x}{\cos x}$ and the second fraction by $\dfrac{1+\sin x}{1+\sin x}$.
$\dfrac{\cos^2 x}{(1+\sin x)\cos x}+\dfrac{1+2\sin x+\sin^2 x}{(1+\sin x)\cos x}=2\sec x$	Simplify. $(1+\sin x)(1+\sin x)=1+\sin x+\sin x+\sin^2 x$ $(1+\sin x)(1+\sin x)=2+2\sin x+\sin^2 x$
$\dfrac{\cos^2 x+1+2\sin x+\sin^2 x}{(1+\sin x)\cos x}=2\sec x$	Add the numerators.
$\dfrac{\cos^2 x+\sin^2 x+1+2\sin x}{(1+\sin x)\cos x}=2\sec x$	Rearrange the terms using the properties of equality.
$\dfrac{1+1+2\sin x}{(1+\sin x)\cos x}=2\sec x$ $\dfrac{2+2\sin x}{(1+\sin x)\cos x}=2\sec x$	Use the Pythagorean Identity $\cos^2\theta+\sin^2\theta=1$ to simplify the numerator.
$\dfrac{2(1+\sin x)}{(1+\sin x)\cos x}=2\sec x$	Factor a two from both terms in the numerator.

$\frac{(1+\sin x)}{(1+\sin x)}\cdot\frac{2}{\cos x}=2\sec x$	Rearrange the terms using the properties of equality.
$1\cdot\frac{2}{\cos x}=2\sec x$ $2\cdot\frac{1}{\cos x}=2\sec x$	Simplify. Rearrange the terms using the properties of equality. Rewrite the fraction using a reciprocal identity.
$2\cdot\sec x=2\sec x$	

Practice Exercises A

Prove the trigonometric identities.

1. $\sec x\cot x=\csc x$
2. $\sin x\sec x=\tan x$
3. $\tan(-x)\cos x=-\sin x$
4. $\cot(-x)\sin x=-\cos x$
5. $\csc x-\sin x=\cot x\cos x$
6. $\tan x\csc x\cos x=1$
7. $\cot x\sec x\sin x=1$
8. $\frac{\tan x\cot x}{\csc x}=\sin x$
9. $\sin^2 x\left(1+\cot^2 x\right)=1$
10. $\sin x\tan x=\frac{1-\cos^2 x}{\cos x}$
11. $\cos x\cot x=\frac{1-\sin^2 x}{\sin x}$
12. $\sec^2 x\csc^2 x=\sec^2 x+\csc^2 x$
13. $\sec x-\sec x\sin^2 x=\cos x$
14. $\csc x-\csc x\cos^2 x=\sin x$
15. $2\sec x=\frac{\cos x}{1-\sin x}+\frac{1-\sin x}{\cos x}$
16. $(\sec x-\tan x)^2=\frac{1-\sin x}{1+\sin x}$

Example 6:
Find all values of *x* if $\cos x - \cos x \sin x = 0$ and $0 \le x \le 360°$.

Answer:

$\cos x - \cos x \sin x = 0$		
$\cos x(1 - \sin x) = 0$		Factor $\cos x$ out of both terms.
$\cos x = 0$	$1 - \sin x = 0$ $1 = \sin x$	Use the zero product property to set each factor equal to zero.
$x = \cos^{-1}(0)$	$\sin^{-1}(1) = x$	To find *x*, an angle, use the inverse cosine and sine.
$x = 90°, 270°$	$x = 90°$	Find all the angles between $0°$ and $360°$ that have a cosine of 0 or a sine of 1.
The angles that satisfy the trigonometric equation are $x = 90°$ and $x = 270°$.		

Example 7:
Find all values of *x* if $2\cos^2 x + \cos x - 1 = 0$ and $0 \le x \le 360°$.

Answer:

$2\cos^2 x + \cos x - 1 = 0$		
$(2\cos x - 1)(\cos x + 1) = 0$		Factor.
$2\cos x - 1 = 0$ $2\cos x = 1$ $\cos x = \frac{1}{2}$	$\cos x + 1 = 0$ $\cos x = -1$	Use the zero product property to set each factor equal to zero.
$x = \cos^{-1}\left(\frac{1}{2}\right)$	$x = \cos^{-1}(-1)$	To find *x*, an angle, use the inverse cosine.
$x = 60°, 300°$	$x = 180°$	Find all the angles between $0°$ and $360°$ that have a cosine of $\frac{1}{2}$ or a cosine of -1.
The angles that satisfy the trigonometric equation are $x = 60°$, $x = 300°$, and $x = 180°$.		

Practice Exercises B

Find all values of *x* if $0 \le x \le 360°$.

1. $2\cos x \sin x - \cos x = 0$
2. $\sqrt{2}\tan x \cos x - \tan x = 0$
3. $\tan x \sin^2 x = \tan x$
4. $\sin x \tan^2 x = \sin x$
5. $\tan^2 x = 3$
6. $2\sin^2 x = 1$
7. $4\cos^2 x - 4\cos x + 1 = 0$
8. $2\sin^2 x + 3\sin x + 1 = 0$
9. $\sin^2 x - 2\sin x = 0$
10. $3\sin x = 2\cos^2 x$

Unit 5 Cluster 6 Honors (F.TF.9)
Prove and Apply Trigonometric Identities

H.5.8 Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

H.5.9 Justify half-angle and double-angle theorems for trigonometric values.

It is possible to find the exact sine, cosine, and tangent values of angles that do not come from special right triangles, but you have to use the angles from the unit circle to find them. The following formulas can be used to find the sine, cosine, and tangent values of angles that are not on the unit circle.

The Cosine of the Sum of Two Angles $\cos(A+B)=\cos A\cos B-\sin A\sin B$	**The Cosine of the Difference of Two Angles** $\cos(A-B)=\cos A\cos B+\sin A\sin B$
The Sine of the Sum of Two Angles $\sin(A+B)=\sin A\cos B+\cos A\sin B$	**The Sine of the Difference of Two Angles** $\sin(A-B)=\sin A\cos B-\cos A\sin B$
The Tangent of the Sum of Two Angles $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$	**The Tangent of the Difference of Two Angles** $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

<u>Proof of the Cosine Difference Formula:</u> $\cos(A-B)=\cos A\cos B+\sin A\sin B$

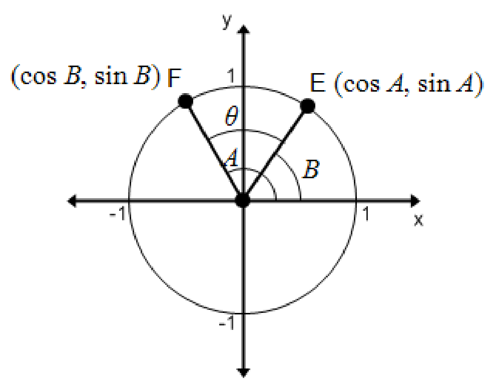


Figure 1

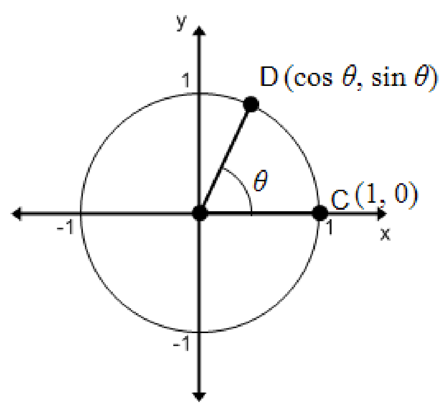


Figure 2

Figure 2 shows an angle in standard position. Figure 1 shows the same angle, but it has been rotated and $\theta=A-B$. The chords opposite the angle, θ, have equal length in both circles. Therefore, $\overline{CD}$ is equal to the length of $\overline{EF}$. Find the measure of $\overline{CD}$ and $\overline{EF}$.

Finding CD

$CD=\sqrt{(\cos\theta-1)^2+(\sin\theta-0)^2}$	Use the distance formula to find the distance between the points $(\cos\theta,\sin\theta)$ and $(1,0)$.
$CD=\sqrt{(\cos^2\theta-2\cos\theta+1)+\sin^2\theta}$	Expand $(\cos\theta-1)^2$. Remember that $(\cos\theta-1)^2=(\cos\theta-1)(\cos\theta-1)$.
$CD=\sqrt{\cos^2\theta+\sin^2\theta-2\cos\theta+1}$	Rearrange the terms so that $\cos^2\theta$ and $\sin^2\theta$ are next to each other.
$CD=\sqrt{1-2\cos\theta+1}$	Recall that $\cos^2\theta+\sin^2\theta=1$ (Pythagorean Identity).
$CD=\sqrt{2-2\cos\theta}$	Simplify.

Finding EF

$EF=\sqrt{(\cos A-\cos B)^2+(\sin A-\sin B)^2}$	Use the distance formula to find the distance between the points $(\cos A,\sin A)$ and $(\cos B,\sin B)$.
$EF=\sqrt{(\cos^2 A-2\cos A\cos B+\cos^2 B)+(\sin^2 A-2\sin A\sin B+\sin^2 B)}$	Expand $(\cos A-\cos B)^2$ and $(\sin A-\sin B)^2$.
$EF=\sqrt{\cos^2 A+\sin^2 A+\cos^2 B+\sin^2 B-2\cos A\cos B-2\sin A\sin B}$	Rearrange the terms so that $\cos^2 A$ and $\sin^2 A$ and $\cos^2 B$ and $\sin^2 B$ are next to each other.
$EF=\sqrt{1+1-2\cos A\cos B-2\sin A\sin B}$	Recall that $\cos^2\theta+\sin^2\theta=1$ (Pythagorean Identity).
$EF=\sqrt{2-2\cos A\cos B-2\sin A\sin B}$	Simplify.

Setting $CD = EF$

$\sqrt{2-2\cos\theta} = \sqrt{2-2\cos A\cos B - 2\sin A\sin B}$	Set $CD = EF$.
$2-2\cos\theta = 2-2\cos A\cos B - 2\sin A\sin B$	Square each side to eliminate the square root.
$-2\cos\theta = -2\cos A\cos B - 2\sin A\sin B$	Subtract 2 from each side of the equation.
$\cos\theta = \cos A\cos B + \sin A\sin B$	Divide each term on both sides of the equation by -2.
$\cos(A-B) = \cos A\cos B + \sin A\sin B$	Recall that $\theta = A-B$

The following identities are needed in order to prove the sine of a sum identity.

Negative Angle Identities
$\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$ $\qquad$ $\tan(-\theta) = -\tan\theta$

<u>Proof of the Cosine of a Sum:</u> $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A+B) = \cos(A-(-B))$	Rewrite the expression so that it is a difference.
$\cos(A+B) = \cos A\cos(-B) + \sin A\sin(-B)$	Use the cosine of a difference identity to rewrite the expression.
$\cos(A+B) = \cos A\cos B + \sin A(-\sin B)$	Use the negative angle identities to eliminate $-B$.
$\cos(A+B) = \cos A\cos B - \sin A\sin B$	Use the commutative property of multiplication to rearrange the terms.

The following identities are needed in order to prove the sine of a sum identity.

Cofunction Identities
$\sin(90°-\theta) = \cos\theta$ $\qquad$ $\cos(90°-\theta) = \sin\theta$ $\qquad$ $\tan(90°-\theta) = \cot\theta$

<u>Proof of the Sine of a Sum:</u> $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A+B) = \cos(90°-(A+B))$	Use the cofunction identity of sine to rewrite the expression in terms of cosine.

$\sin(A+B)=\cos(90°-A-B)$ $\sin(A+B)=\cos((90°-A)-B)$	Distribute the negative and then group the first two terms.
$\sin(A+B)=\cos(90°-A)\cos(B)+\sin(90°-A)\sin(B)$	Use the cosine of a difference identity to rewrite the expression.
$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$	Use the cofunction identities to rewrite the expression.

Proof of the Sine of a Difference: $\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$

$\sin(A-B)=\sin(A+(-B))$	Rewrite the expression so that it is a sum.
$\sin(A-B)=\sin(A)\cos(-B)+\cos(A)\sin(-B)$	Use the sine of sum identity to rewrite the expression.
$\sin(A-B)=\sin(A)\cos(B)+\cos(A)(-\sin B)$	Use the negative angle identities to eliminate $-B$.
$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin B$	Use the commutative property of multiplication to rewrite the expression.

Proof of the Tangent of a Sum Identity: $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$	Use the definition of tangent to rewrite the expression.
$\tan(A+B)=\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$	Use the sine and cosine of a sum identities to rewrite the expressions.
$\tan(A+B)=\dfrac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$	Divide each term in the numerator and the denominator by $\cos A\cos B$.
$\tan(A+B)=\dfrac{\dfrac{\sin A}{\cos A}\cdot\dfrac{\cos B}{\cos B}+\dfrac{\cos A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}{\dfrac{\cos A}{\cos A}\cdot\dfrac{\cos B}{\cos B}-\dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}$	Rewrite each expression using properties of equality.
$\tan(A+B)=\dfrac{\tan A\cdot 1+1\cdot\tan B}{1\cdot 1-\tan A\cdot\tan B}$	Simplify using the fact that $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$.
$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$	Simplify.

Proof of the Tangent of a Difference Identity: $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$\tan(A-B)=\dfrac{\sin(A-B)}{\cos(A-B)}$	Use the definition of tangent to rewrite the expression.
$\tan(A-B)=\dfrac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$	Use the sine and cosine of a difference identities to rewrite the expressions.
$\tan(A-B)=\dfrac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}+\dfrac{\sin A\sin B}{\cos A\cos B}}$	Divide each term in the numerator and the denominator by $\cos A\cos B$.
$\tan(A-B)=\dfrac{\dfrac{\sin A}{\cos A}\cdot\dfrac{\cos B}{\cos B}-\dfrac{\cos A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}{\dfrac{\cos A}{\cos A}\cdot\dfrac{\cos B}{\cos B}+\dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}$	Rewrite each expression using properties of equality.
$\tan(A-B)=\dfrac{\tan A\cdot 1-1\cdot\tan B}{1\cdot 1+\tan A\cdot\tan B}$	Simplify using the fact that $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$.
$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$	Simplify.

Example 1:

Find the exact value of $\sin 165°$.

Answer:

$\sin 165°=\sin(30°+135°)$	Find two special angles that add or subtract to $165°$. (There are several possibilities.)
$\sin 165°=\sin 30°\cos 135°+\cos 30°\sin 135°$	Use the sine of a sum identity to rewrite the expression.
$\sin 165°=\left(\dfrac{1}{2}\right)\left(-\dfrac{\sqrt{2}}{2}\right)+\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right)$	Substitute known values.
$\sin 165°=-\dfrac{\sqrt{2}}{4}+\dfrac{\sqrt{6}}{4}$ $\sin 165°=\dfrac{-\sqrt{2}+\sqrt{6}}{4}$	Simplify.

Example 2:
Find the exact value of $\tan 165°$.

Answer:

$\tan 165° = \tan(210° - 45°)$	Find two special angles that add or subtract to $165°$. (There are several possibilities.)
$\tan 165° = \dfrac{\tan 210° - \tan 45°}{1 + \tan 210° \tan 45°}$	Use the tangent of a difference identity to rewrite the expression.
$\tan 165° = \dfrac{\left(\frac{\sqrt{3}}{3}\right) - (1)}{1 + \left(\frac{\sqrt{3}}{3}\right)(1)}$	Substitute known values.
$\tan 165° = \dfrac{\frac{\sqrt{3}}{3} - \frac{3}{3}}{\frac{3}{3} + \frac{\sqrt{3}}{3}}$ $\tan 165° = \dfrac{\frac{\sqrt{3} - 3}{3}}{\frac{3 + \sqrt{3}}{3}}$ $\tan 165° = \dfrac{\sqrt{3} - 3}{3 + \sqrt{3}}$	Simplify.

Practice Exercises A

Use a sum or difference formula to find an exact value.

1.	$\sin 15°$	2.	$\cos 15°$	3.	$\tan 75°$
4.	$\sin 75°$	5.	$\cos 105°$	6.	$\tan 105°$
7.	$\sin 195°$	8.	$\cos 195°$	9.	$\tan 255°$
10.	$\sin 255°$	11.	$\cos 285°$	12.	$\tan 285°$

Double and Half Angle Formulas

Double-Angle Theorems

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

<u>Justification of double angle theorem for sine:</u>

$\sin 2\theta = \sin\left(\theta+\theta\right)$	Substitute $2\theta = \theta + \theta$.
$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$	Use the sine of a sum formula.
$\sin 2\theta = 2\sin\theta\cos\theta$	Simplify.

<u>Justification of double angle theorem for cosine:</u>

$\cos 2\theta = \cos\left(\theta+\theta\right)$	Substitute $2\theta = \theta + \theta$.
$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta$	Use the cosine of a sum formula.
$\sin 2\theta = \cos^2\theta - \sin^2\theta$	Simplify.

<u>Justification of double angle theorem for tangent:</u>

$\tan 2\theta = \tan\left(\theta+\theta\right)$	Substitute $2\theta = \theta + \theta$.
$\tan 2\theta = \dfrac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta}$	Use the tangent of a sum formula.
$\tan 2\theta = \dfrac{2\tan\theta\tan\theta}{1-\tan^2\theta}$	Simplify.

Half-Angle Theorems

$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

<u>Justification of half-angle theorem for sine:</u>

$\cos 2\theta = 1-2\sin^2\theta$	Use $\cos 2\theta = 1-2\sin^2\theta$. Solve the double angle formula for $\sin\theta$.
$\cos 2\theta + 2\sin^2\theta = 1$	
$2\sin^2\theta = 1-\cos 2\theta$	
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$	
$\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$	
$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{2}}$	Substitute $\theta=\frac{\alpha}{2}$.
$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$	Simplify.

<u>Justification of half-angle theorem for cosine:</u>

$\cos 2\theta = 2\cos^2\theta-1$	Use $\cos 2\theta = 2\cos^2\theta-1$. Solve the double angle formula for $\cos\theta$.
$\cos 2\theta - 2\cos^2\theta = -1$	
$-2\cos^2\theta = -1-\cos 2\theta$	
$\cos^2\theta = \frac{1+\cos 2\theta}{2}$	
$\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$	
$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2}}$	Substitute $\theta=\frac{\alpha}{2}$.
$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$	Simplify.

<u>Justification of half-angle theorem for tangent:</u>

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$ Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Substitute $\theta = \dfrac{\alpha}{2}$.

$$\tan\frac{\alpha}{2} = \pm\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}}$$ Use the half-angle theorems for sine and cosine.

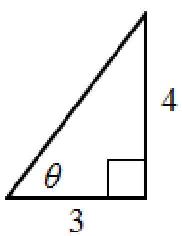
$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{\frac{1-\cos\alpha}{2}}{\frac{1+\cos\alpha}{2}}}$$

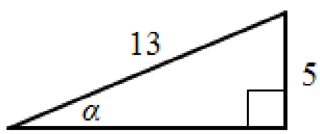
$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{\frac{1}{2}(1-\cos\alpha)}{\frac{1}{2}(1+\cos\alpha)}}$$ Simplify using properties of radicals and exponents.

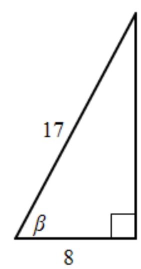
$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Practice Exercises B

Use the figures to find the exact value of each trigonometric function.







1. $\sin 2\theta$	2. $\cos 2\theta$	3. $\tan 2\theta$
4. $\sin 2\alpha$	5. $\cos 2\alpha$	6. $\tan 2\alpha$
7. $\sin 2\beta$	8. $\cos 2\beta$	9. $\tan 2\beta$
10. $\sin\frac{\theta}{2}$	11. $\cos\frac{\theta}{2}$	12. $\tan\frac{\theta}{2}$
13. $\sin\frac{\alpha}{2}$	14. $\cos\frac{\alpha}{2}$	15. $\tan\frac{\alpha}{2}$
16. $\sin\frac{\beta}{2}$	17. $\cos\frac{\beta}{2}$	18. $\tan\frac{\beta}{2}$

Unit 6

Circles With and Without Coordinates

Unit 6 Cluster 1 (G.C.1, G.C.2, G.C.3, and Honors G.C.4)
Understand and Apply Theorems about Circles

Cluster 1: Understanding and applying theorems about circles

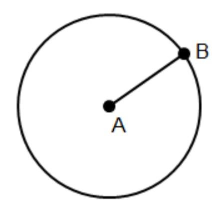
6.1.1 Prove that all circles are similar.
6.1.2 Understand relationships with inscribed angles, radii, and chords (the relationship between central, inscribed, circumscribed; the relationship between inscribed angles on a diameter; the relationship between radius and the tangent).
6.1.3 Construct the inscribed and circumscribed sides of a triangle.
6.1.3 Prove properties of angles for a quadrilateral inscribed in a circle.
6.1.4 (**Honors**) Construct a tangent line from a point outside a given circle to a circle.

VOCABULARY

A **circle** is the set of all points equidistant from a given point which is called the **center of the circle**.

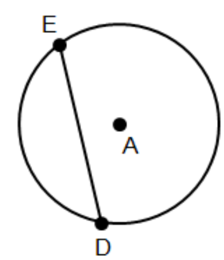
A **radius** is any segment with endpoints that are the center of the circle and a point on the circle. **Radii** is the plural of radius.

$\overline{AB}$ is the radius of circle A. The center of the circle is point A.



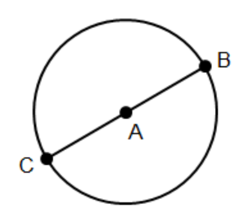
A segment with endpoints on the circle is called a **chord**.

$\overline{DE}$ is a chord of circle A.



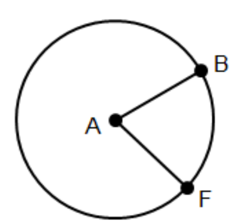
A **diameter** is any chord with endpoints that are on the circle and that passes through the center of the circle. The diameter is the longest chord of a circle.

$\overline{CB}$ is the diameter of circle A.



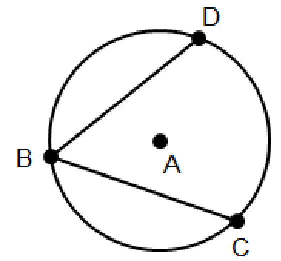
An angle that intersects a circle in two points and that has its vertex at the center of the circle is a **central angle**.

$\angle BAF$ is a central angle of circle A.



An angle that intersects a circle in two points and that has its vertex on the circle is an **inscribed angle**.

$\angle DBC$ *is an inscribed angle of circle A.*

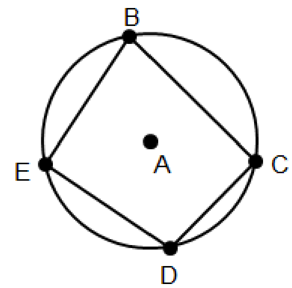


A polygon that is **circumscribed by a circle** has all of its vertices on the circle and the polygon's interior is completely contained within the circle.

Circle A is circumscribed about Quadrilateral BCDE.

A planar shape or solid completely enclosed by (fits snugly inside) another geometric shape or solid is an **inscribed figure**. Each of the vertices of the enclosed figure must lie on the "outside" figure.

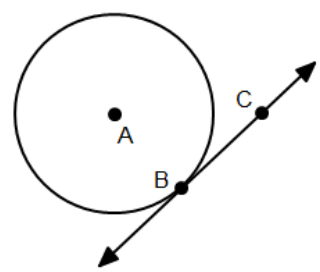
Quadrilateral BCDE is inscribed in circle A.



A line that intersects a circle in only one point is a **tangent line**.

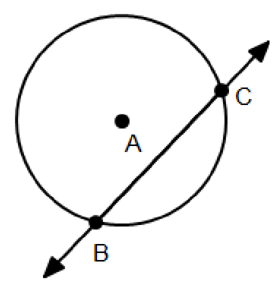
The point where the tangent line and the circle intersect is the **point of tangency**.

$\overleftrightarrow{BC}$ *is a tangent line to circle A. Point B is the point of tangency.*



A line that intersects a circle in two points is a **secant line**.

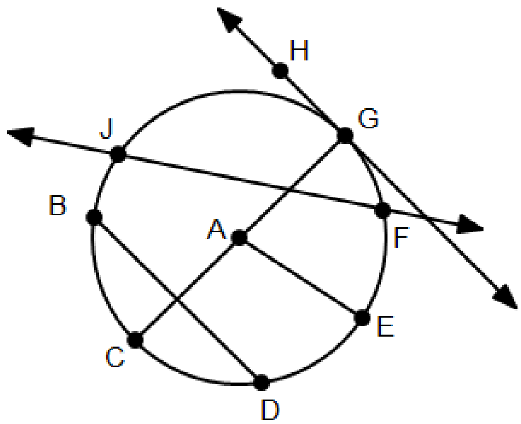
$\overleftrightarrow{BC}$ *is a secant line to circle A.*



Other helpful sources: http://www.mathgoodies.com/lessons/vol2/geometry.html

Practice Exercises A

Identify a chord, tangent line, diameter, two radii, the center, and point of tangency and, a central angle.



1. chord: ______________________________
2. tangent line: _________________________
3. diameter: ____________________________
4. radius: ______________________________
5. point of tangency : ____________________
6. center: ______________________________
7. central angle: ________________________

Identify the term that best describes the given line, segment, or point.

8. $\overline{AF}$

9. $\overline{PF}$

10. C

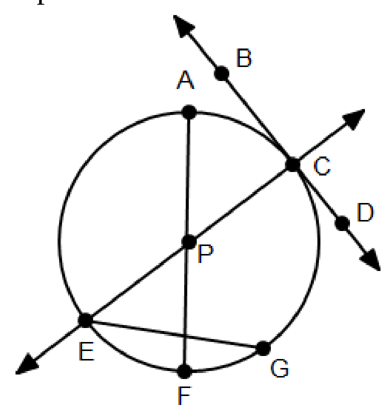
11. $\overleftrightarrow{BD}$

12. $\overline{EG}$

13. $\overline{PG}$

14. $\overleftrightarrow{CE}$

15. P

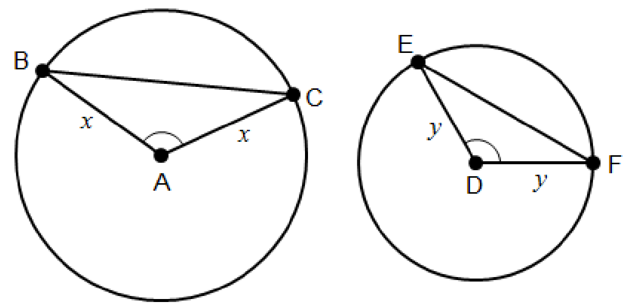


Prove All Circles are Similar

Figures that are similar have corresponding parts that are proportional. To prove that all circles are similar, you will need to prove that their corresponding parts are proportional.

Proof that all circles are Similar

Given: $\angle A \cong \angle D$, radius of circle A is *x*, and radius of circle D is *y*.
Prove: Circle A is similar to Circle D.



First we need to prove that $\triangle ABC \sim \triangle DEF$ so we need to establish AA criterion. Since $\overline{AB} \cong \overline{AC}$, $\triangle ABC$ is an isosceles triangle. The base angles of an isosceles triangle are congruent to one another. Therefore $\angle B \cong \angle C$. Similarly, $\overline{DE} \cong \overline{DF}$ so $\triangle DEF$ is also an isosceles triangle. Therefore $\angle E \cong \angle F$. The sum of the angles in a triangle is $180°$ so $\angle A + \angle B + \angle C = 180°$ and $\angle D + \angle E + \angle F = 180°$. We know that $\angle B \cong \angle C$ and $\angle E \cong \angle F$ so $\angle A + 2\angle B = 180°$ and $\angle D + 2\angle E = 180°$. Solving each equation for $\angle A$ and $\angle D$ yields $\angle A = 180° - 2\angle B$ and $\angle D = 180° - 2\angle E$. Since $\angle A \cong \angle D$ we know that $180° - 2\angle B = 180° - 2\angle E$ and $\angle B \cong \angle E$. By AA similarity $\triangle ABC \sim \triangle DEF$. Because the two triangles are similar their sides will be proportional. The ratio of proportionality is $\frac{\overline{AB}}{\overline{DE}} = \frac{x}{y}$. $\overline{AB}$ is a radius of circle A and $\overline{DE}$ is a radius of circle D. The ratio of the radii of the circles is $\frac{x}{y}$. The diameter of circle A is $2x$ and the diameter of circle D is $2y$. The ratio of the diameter of circle A to the diameter of circle D is $\frac{2x}{2y} = \frac{x}{y}$. The circumference of circle A is $2\pi x$ and the circumference of circle D is $2\pi y$. The ratio of the circumference of circle A to the circumference of circle D is $\frac{2\pi x}{2\pi y} = \frac{x}{y}$. The corresponding parts of circle A are proportional to the corresponding parts of circle D, therefore circle A is similar to circle D.

In addition to having corresponding parts proportional to one another, figures that are similar to one another are dilations of one another. A dilation is a transformation that produces an image that is the same shape as the original figure but the image is a different size. The dilation uses a center and a scale factor to create a proportional figure. The ratio of the corresponding parts is the scale factor of the dilation.

Practice Exercises B

1. Given a circle of a radius of 3 and another circle with a radius of 5, compare the ratios of the two radii, the two diameters, and the two circumferences.

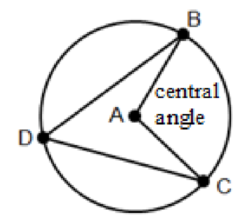
2. Given a circle of a radius of 6 and another circle with a radius of 4, compare the ratios of the two radii, the two diameters, and the two circumferences.

Properties of Central Angles and Inscribed Angles

VOCABULARY

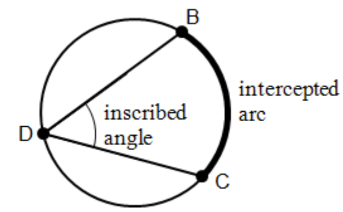
An **arc** is a portion of a circle's circumference.

An **intercepted arc** is the arc that lies in the interior of an angle and has its endpoints on the angle.

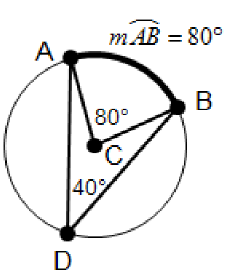


A **central angle** is an angle that intersects a circle in two points and that has its vertex at the center of the circle. The measure of the angle is the same as the measure of its intercepted arc.

An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle. The measure of the angle is half the measure of the intercepted arc.

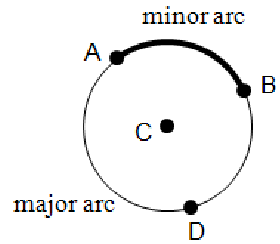


$\angle ACB$ is a central angle of circle C and its intercepted arc is AB and $m\angle ACB = mAB$. $\angle ADB$ is an inscribed angle of circle A and its intercepted arc is AB and $m\angle ADB = \frac{1}{2}mAB$.



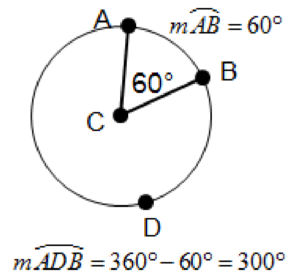
If the $m\angle ACB$ is less than $180°$, then A, B, and all the points on $\odot C$ that lie in the interior of $m\angle ACB$ form a **minor arc**. A minor arc is named by two consecutive points. The measure of a minor arc is the measure of its central angle.

AB is the minor arc of circle C.

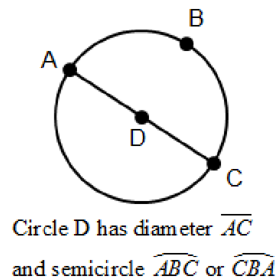


Points A, B, and all points on $\odot C$ that do not lie on AB form a **major arc**. A major arc is named by three consecutive points. The measure of a major arc is 360° minus the measure of the related minor arc.

ADB is the major arc of circle C.

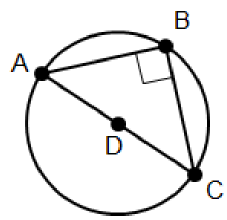


A **semicircle** is an arc whose central angle measures 180°. A semicircle is named by three points.



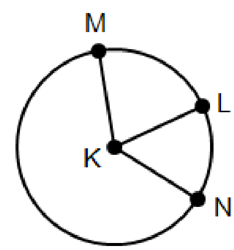
An inscribed angle that intercepts a semicircle is a right angle.

$$m\angle ABC = \frac{1}{2}\left(mAC\right) = \frac{1}{2}(180^\circ) = 90^\circ$$



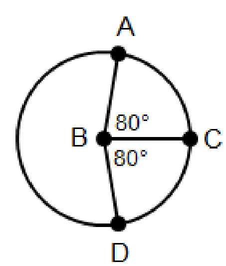
The **arc addition postulate** states that the measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.

$$mML + mLN = mMN$$



Congruent arcs are arcs with the same measure either in the same circle or congruent circles. Congruent central angles or inscribed angles have congruent arcs and congruent arcs have congruent central angles or inscribed angles.

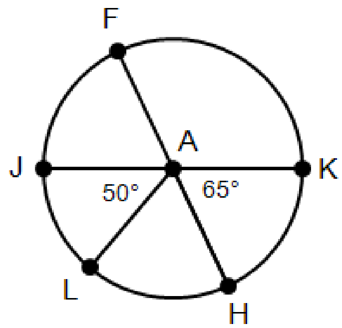
$$\angle ABC \cong \angle CBD \Leftrightarrow AC \cong CD$$



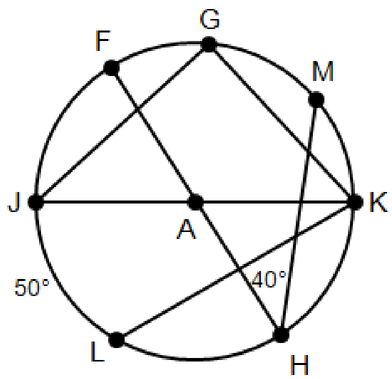
Practice Exercises C

$\overline{FH}$ and $\overline{JK}$ are diameters. Find the measure of each angle or arc.

1. $m\angle FAJ$
2. $m\angle ALH$
3. $m\angle KAF$
4. $m\overset{\frown}{JL}$
5. $m\overset{\frown}{LH}$
6. $m\overset{\frown}{HK}$
7. $m\overset{\frown}{KF}$
8. $m\overset{\frown}{JF}$
9. $m\overset{\frown}{JH}$
10. $m\overset{\frown}{JHF}$

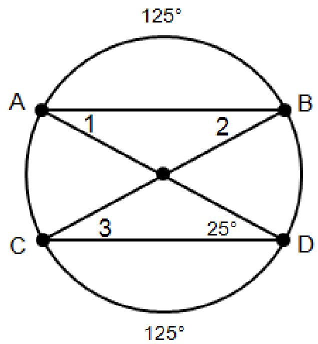


$\overline{FH}$ and $\overline{KJ}$ are diameters, $m\angle FHM = 40°$, and $m\overset{\frown}{JL} = 50°$. Find the measure of each angle or arc.

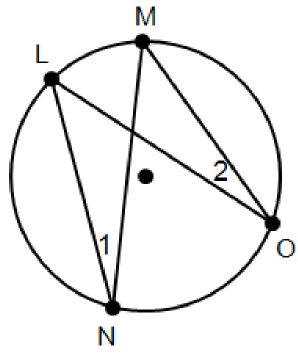


11. $m\overset{\frown}{JF}$
12. $m\overset{\frown}{LH}$
13. $m\angle JKL$
14. $m\overset{\frown}{FM}$
15. $m\angle HAK$
16. $m\overset{\frown}{HK}$
17. $m\overset{\frown}{KF}$
18. $m\overset{\frown}{KH}$
19. $m\overset{\frown}{MK}$
20. $m\overset{\frown}{JGK}$

21. Find the measure of angles 1, 2, and 3.

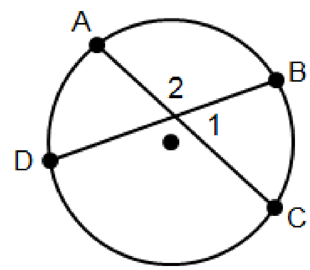


22. Find the measure of angles 1 and 2 if $m\angle 1 = 2x - 13$ and $m\angle 2 = x$.



Theorems

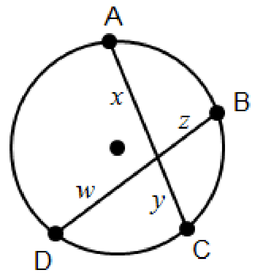
If two chords intersect inside a circle, then the measure of each angle formed is half the sum of the measures of the arcs intercepted by the angle and its vertical angle.



$$m\angle 2 = \frac{1}{2}\left(m\overset{\frown}{CD} + m\overset{\frown}{AB}\right)$$

$$m\angle 1 = \frac{1}{2}\left(m\overset{\frown}{BC} + m\overset{\frown}{AD}\right)$$

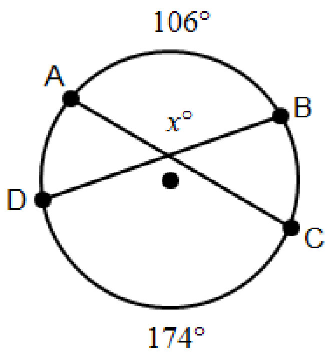
If two chords intersect inside a circle, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.



$$xy = wz$$

Example 1:

Find the value of *x*.



The two chords forming the angle that measures $x°$ intercepts the two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$. The measure of $x°$ will be equal to one-half the sum of the measures of the intercepted arcs.

Answer:

$$x° = \frac{1}{2}\left(m\overset{\frown}{AB} + m\overset{\frown}{CD}\right)$$

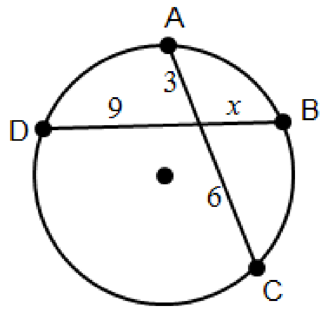
$$x° = \frac{1}{2}(106° + 174°)$$

$$x° = \frac{1}{2}(280°)$$

$$x° = 140°$$

Example 2:

Find the value of *x*.



The chords *AC* and *BD* intersect inside the circle, therefore the product of the lengths of the segments of each chord are equal to one another. By setting up this equation you can solve for *x*.

Answer:

$$9x = 3 \cdot 6$$

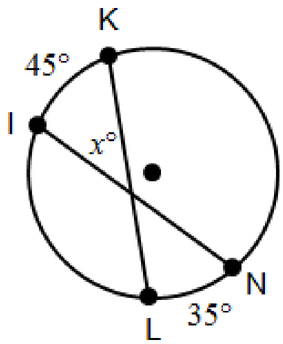
$$9x = 18$$

$$x = 2$$

Practice Exercises D

Find the value of x.

1.



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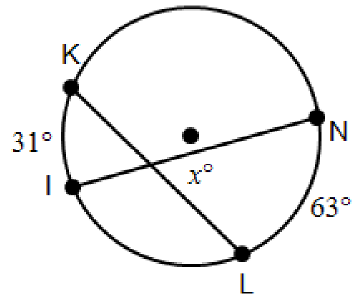

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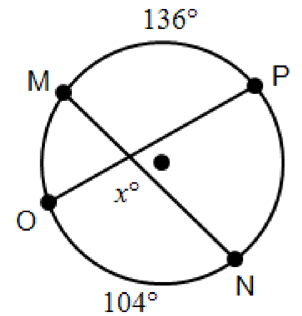

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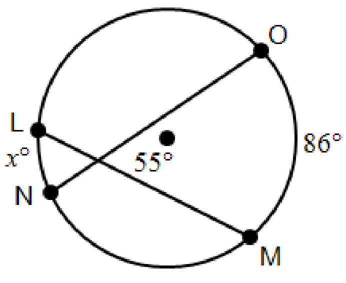

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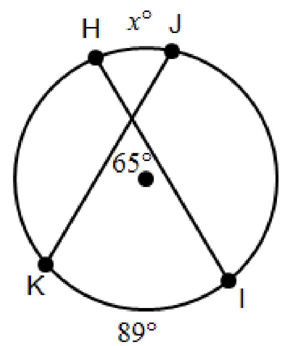

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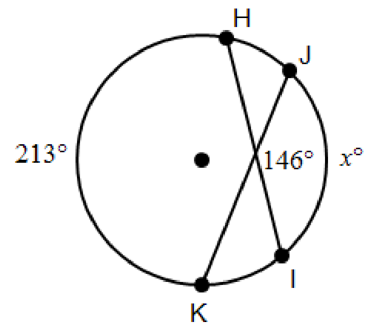

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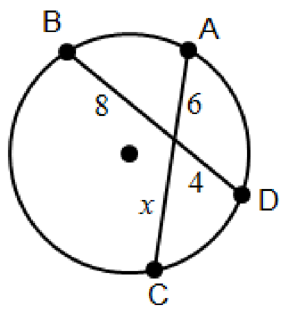

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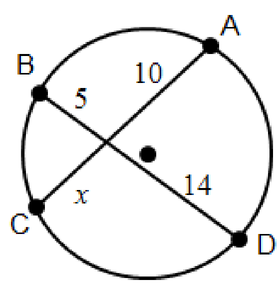

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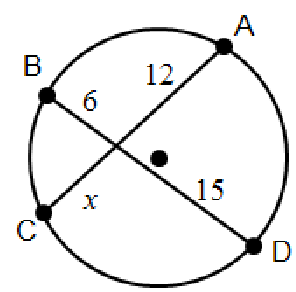

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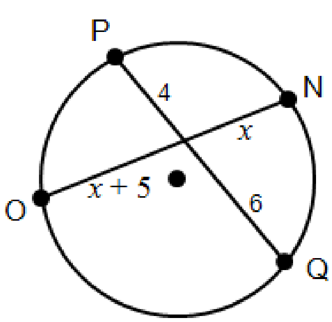

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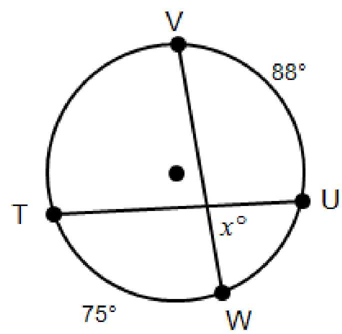

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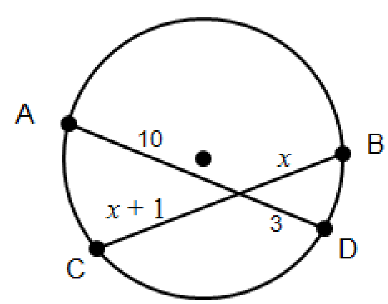


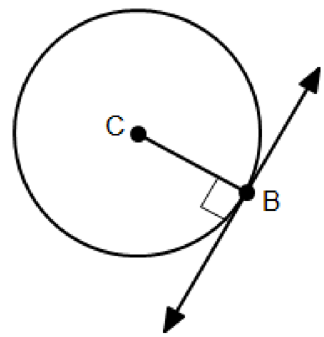
Theorems

If a line is tangent to a circle, then it is perpendicular to the radius drawn at the point of tangency.

If l is tangent to $\odot C$ at B, then $l \perp$ to $\overline{CB}$.

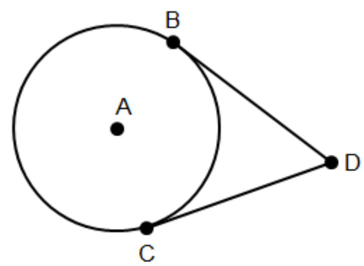
In a plane, if a line is perpendicular to a radius of a circle at its endpoints on the circle, then the line is tangent to the circle.

If $l \perp$ to $\overline{CB}$, then l is tangent to $\odot C$ at B.



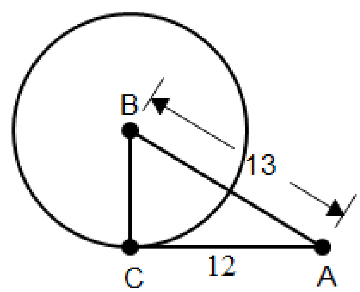
If two segments from the same point outside a circle are tangent to the circle, then they are congruent.

If $\overline{BD}$ and $\overline{CD}$ are tangent to circle A at points B and C, then $\overline{BD} \cong \overline{CD}$.



Example 3: Using Properties of Tangents

$\overline{AC}$ is tangent to $\odot B$ at point C. Find BC.

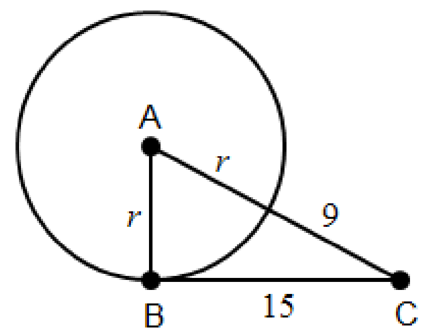


Answer:

$(AC)^2 + (BC)^2 = (AB)^2$ $12^2 + (BC)^2 = 13^2$ $144 + (BC)^2 = 169$ $(BC)^2 = 169 - 144$ $(BC)^2 = 25$ $BC = 5$	Since $\overline{AC}$ is tangent to circle B at point C, $\overline{AC} \perp \overline{BC}$. This makes ΔABC a right triangle and the Pythagorean Theorem can be used to find BC.

Example 4: Finding the Radius of a Circle

$\overline{BC}$ is tangent to circle A at point B. Solve for *r*.

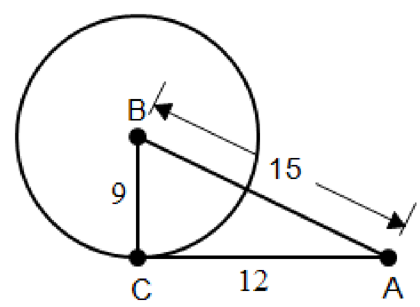


Answer:

$(BC)^2 + (AB)^2 = (AC)^2$ $r^2 + 15^2 = (r+9)^2$ $r^2 + 225 = r^2 + 18r + 81$ $225 = 18r + 81$ $225 - 81 = 18r$ $144 = 18r$ $8 = r$	Since $\overline{BC}$ is tangent to circle A at point B, $\overline{AB} \perp \overline{BC}$. This makes ΔABC a right triangle and the Pythagorean Theorem can be used to find *r*.

Example 5: Verify a Line is Tangent to a Circle

Show that $\overline{AC}$ is tangent to $\odot B$.

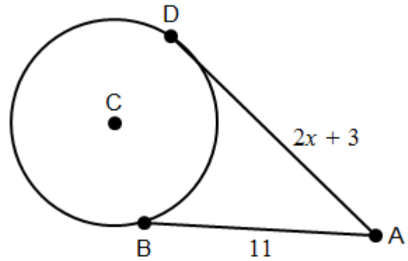


Answer:

$9^2 + 12^2 = 15^2$ $81 + 144 = 225$ $225 = 225$	If you can show that $9^2 + 12^2 = 15^2$, then the lines are perpendicular. 225 = 225 is a true statement, so the lines are perpendicular.

Example 6: Using Properties of Tangents

Find x if $\overline{AB}$ is tangent to circle C at point B and $\overline{AD}$ is tangent to circle C at point D.



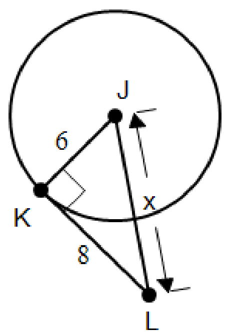
Answer:

$AD = AB$ $2x+3=11$ $2x=8$ $x=4$	Because $\overline{AB}$ and $\overline{AD}$ are tangent to the same circle and contain the same exterior point, the measures of each length are equal.

Practice Exercises E

Find the value of x.

1.



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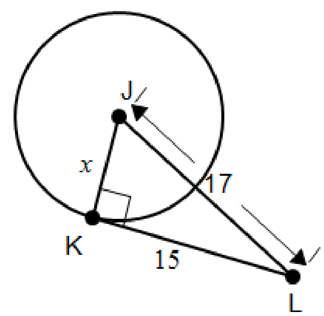

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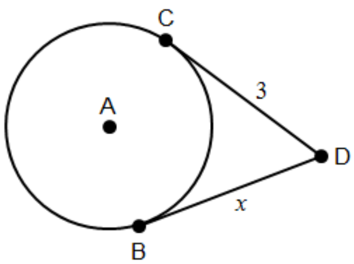
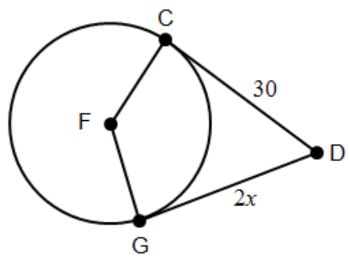
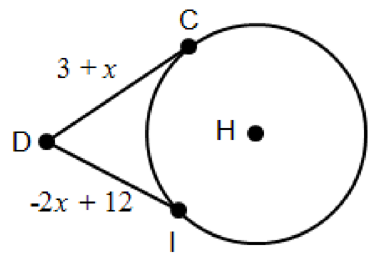


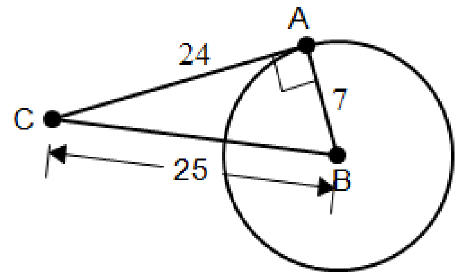
5.



6.



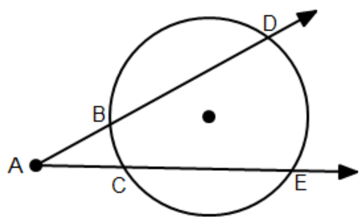
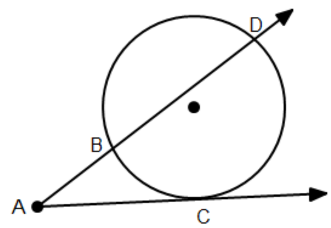
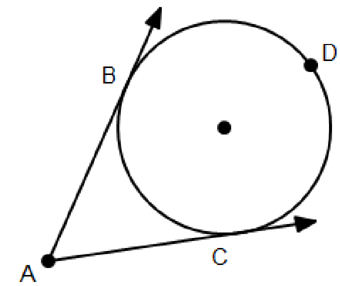
7. Prove that radius $\overline{AB} \perp \overline{AC}$ using the Pythagorean Theorem.



Circumscribed Angles

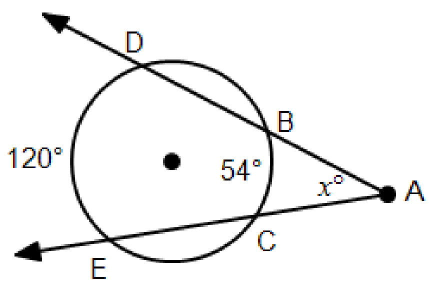
Theorem

If two secants, a secant and a tangent, or two tangents intersect in the exterior of a circle, then the measure of the angle formed is one-half the positive difference of the measures of the intercepted arcs.

Two Secants	Secant and Tangent	Two Tangents
		
$m\angle A = \frac{1}{2}\left(mDE - mBC\right)$	$m\angle A = \frac{1}{2}\left(mDC - mBC\right)$	$m\angle A = \frac{1}{2}\left(mCDB - mBC\right)$

Example 7:

Find the value of *x*.



$\angle A$ is formed by two secants so its measure will be one-half the positive difference of the two intercepted arcs DE and BC.

$$m\angle A = \frac{1}{2}\left(mDE - mBC\right)$$

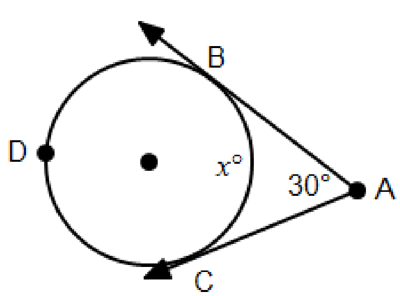
$$x° = \frac{1}{2}\left(120° - 54°\right)$$

$$x° = \frac{1}{2}\left(66°\right)$$

$$x° = 33°$$

Example 8:

Find the value of *x*.



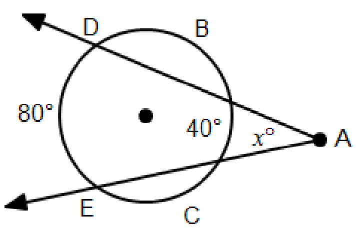
$\angle A$ is formed by two tangents so its measure will be one-half the positive difference of the two intercepted arcs $BDC = 360° - x°$ and $BC = x°$.

	$m\angle A = \frac{1}{2}\left(mBDC - mBC\right)$ $30^\circ = \frac{1}{2}\left(\left(360^\circ - x^\circ\right) - x^\circ\right)$ $30^\circ = \frac{1}{2}\left(360^\circ - 2x^\circ\right)$ $30^\circ = 180^\circ - x^\circ$ $-150^\circ = -x^\circ$ $150^\circ = x^\circ$

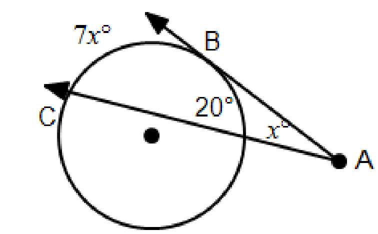
Practice Exercises F

Find the value of x. Assume lines that appear tangent are tangent.

1.



2.



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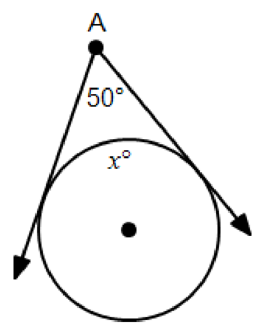

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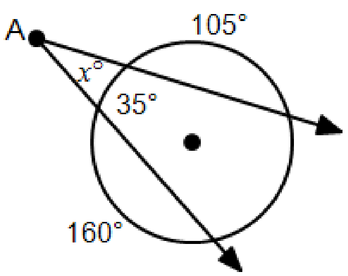

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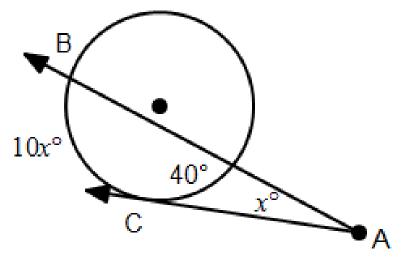

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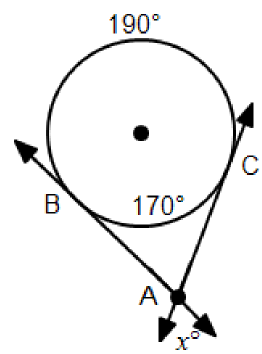


Constructions

See Secondary Math 1 for more help with constructions.

Circumscribe a triangle with a circle

Step 1: Create a triangle.

Step 2: Construct the perpendicular bisector of $\overline{AB}$ by setting the compass to slightly more than half the distance between points A and B.

Step 3: With the compass on point A, draw an arc on both sides of $\overline{AB}$.

Step 4: With the compass on point B, draw an arc on both sides of $\overline{AB}$.

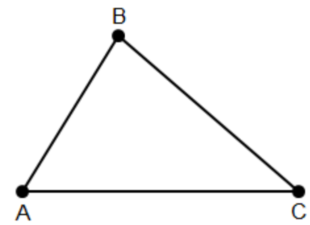
Step 5: With a straightedge, draw the line connecting the intersection points. This is the perpendicular bisector of $\overline{AB}$.

Step 6: Construct the perpendicular bisector of another side of the triangle.

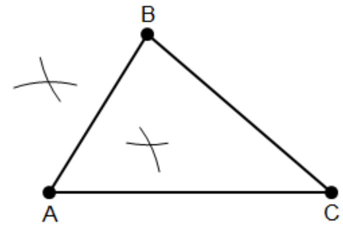
Step 7: The intersection of the two perpendicular bisectors will be equidistant from the vertices. Set the compass to the distance between a vertex and this intersection point.

Step 8: With the compass on the intersection of the perpendicular bisectors, draw a circle that circumscribes the triangle.

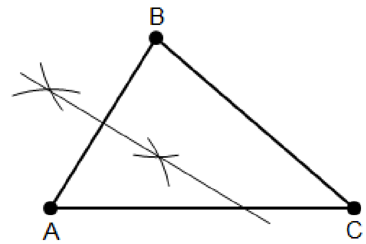
Step 1



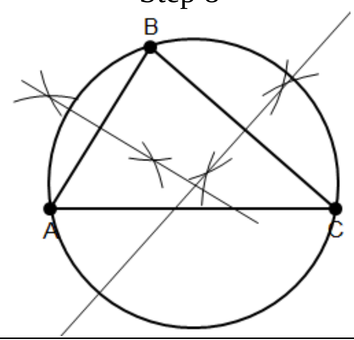
Steps 3 and 4



Step 5



Step 8



Inscribe a circle in a triangle

Step 1: Draw a triangle. Label the vertices A, B, and C.

Step 2: Bisect $\angle A$ of the triangle by setting the compass to a medium length. With the compass somewhere on point A draw an arc from $\overline{AB}$ to $\overline{AC}$.

Step 3: With the compass on $\overline{AB}$ where the arc intersects the segment, draw an arc in the interior of $\angle A$.

Step 4: With the compass on $\overline{AC}$ where the arc intersects the segment, draw an arc in the interior of $\angle A$.

Step 5: Connect point A to the intersection of the two arcs in its interior with a straightedge. This is the bisector of the angle.

Step 6: Bisect another one of the angles.

Step 7: The intersection point of the two bisectors is the center of the inscribed circle.

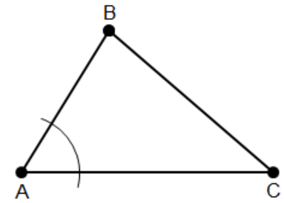
Step 8: Construct a perpendicular from the center point to one side of the triangle.

Step 9: Set the compass to be more than the distance from the center to a side of the triangle $\left(\overline{AC}\right)$. With the compass on the intersection of the two angle bisectors, draw two arcs that intersect a side of the triangle $\left(\overline{AC}\right)$.

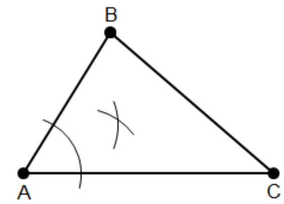
Step 10: Set the compass to the distance between the two arcs that intersect a side of the triangle $\left(\overline{AC}\right)$. With the compass on one of the intersections, draw an arc below a side of the triangle $\left(\overline{AC}\right)$.

Step 11: With the compass on the other intersection, draw an arc below a side of the triangle $\left(\overline{AC}\right)$.

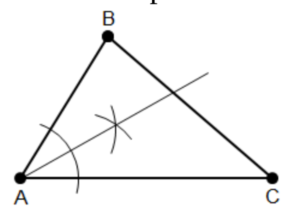
Step 2



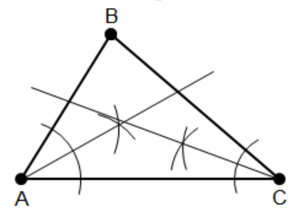
Steps 3 and 4



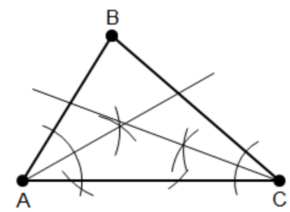
Step 5



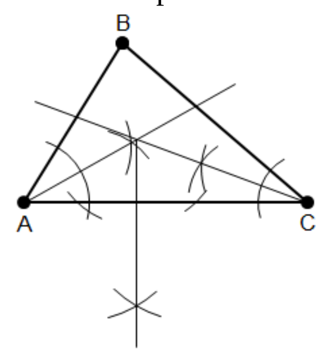
Step 6

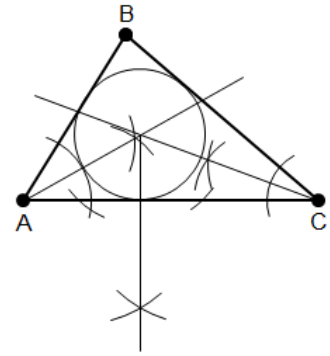


Step 8



Step 12

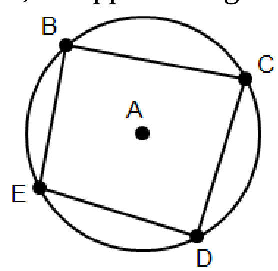


Step 12: Draw the line connecting the intersection of the two angle bisectors to the intersection of the two arcs below a side of the triangle $\left(\overline{AC}\right)$. Step 13: Place the compass on the center point; adjust the length of the compass to touch the intersection of the perpendicular line and the side of the triangle. Step 14: Now create the inscribed circle.	Step 14 

Quadrilaterals Inscribed in Circles

Theorem

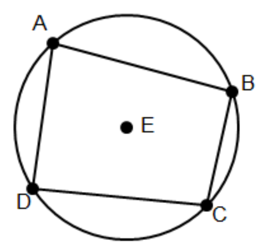
If a quadrilateral is inscribed in a circle, its opposite angles are supplementary.



<u>Proof of Theorem:</u>

Given: Quadrilateral ABCD is inscribed in circle E

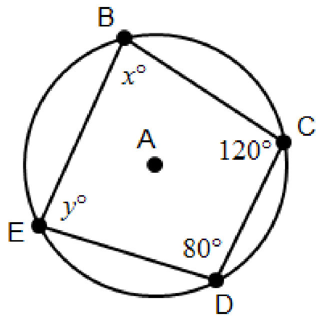
Prove: $\angle A$ and $\angle C$ are supplementary. $\angle B$ and $\angle D$ are supplementary.



By arc addition and the definitions of arc measure, $mBCD + mBAD = 360°$ and $mABC + mADC = 360°$. Since the measure of the intercepted arc is twice the measure of the inscribed angle, $2m\angle A = mBCD$, $2m\angle C = mBAD$, $2m\angle B = mADC$ and $2m\angle D = mABC$. By substitution $2m\angle A + 2m\angle C = 360°$ and $2m\angle B + 2m\angle D = 360°$. Using reverse distribution, the equations can be rewritten as $2\left(m\angle A + m\angle C\right) = 360°$ and $2\left(m\angle B + m\angle D\right) = 360°$. Applying the division property of equality the equations can be rewritten as $m\angle A + m\angle C = 180°$ and $m\angle B + m\angle D = 180°$. By definition of supplementary, $\angle A$ and $\angle C$ are supplementary and $\angle B$ and $\angle D$ are supplementary.

Example 1:

Find x and y.



The opposite angles of a quadrilateral inscribed in a circle are supplementary.

Answer:

$x° = 180° - m\angle EDC$

$x° = 180° - 80°$

$x° = 100°$

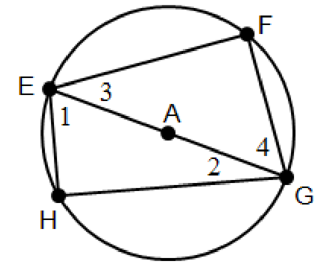
$y° = 180° - m\angle BCD$

$y° = 180° - 120°$

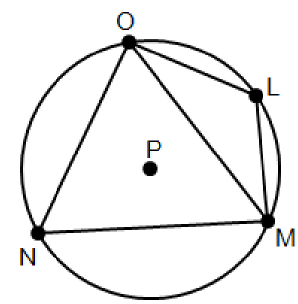
$y° = 60°$

Practice Exercises G

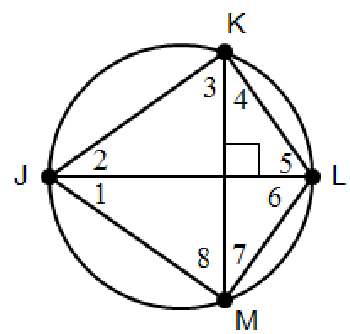
1. Triangles EFG and EGH are inscribed in $\odot I$ with $EF \cong FG$. Find the measure of each numbered angle if $m\angle 1 = 12x - 8$ and $m\angle 2 = 3x + 8$.



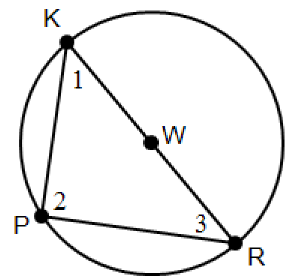
2. Quadrilateral LMNO is inscribed in $\odot P$. If $m\angle M = 80$ and $m\angle N = 40$, then find $m\angle O$ and $m\angle L$.



3. Find the measure of each numbered angle for the figure if $m\overset{\frown}{JK} = 120$.



4. Find the measure of each numbered angle for the figure if $m\angle R = \frac{1}{2}x$ and $m\angle K = \frac{1}{3}x + 5$.



5. Quadrilateral QRST is inscribed in a circle. If $m\angle Q = 45$ and $m\angle R = 100$, find $m\angle S$ and $m\angle T$.

6. Quadrilateral ABCD is inscribed in a circle. If $m\angle C = 28$ and $m\angle B = 110$, find $m\angle A$ and $m\angle D$.

(Honors) Construct a tangent line from a point outside a given circle to a circle

Step 1: Start with a circle with center A, and a point B outside of the circle.

Step 2: Draw a line segment with endpoints A and B.

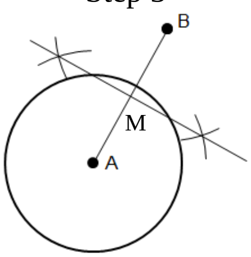
Step 3: Find the midpoint M of $\overline{AB}$ by constructing the perpendicular bisector of $\overline{AB}$. Depending of the size of the circle and the location of point B, the midpoint may be inside or outside of the circle.

Step 4: Place the compass on point M and set the compass width to the center A of the circle.

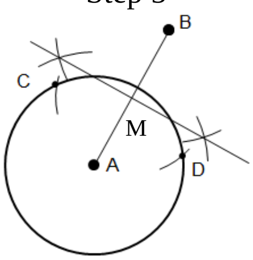
Step 5: Without changing the compass width, draw an arc that intersects the circle in two points. These points (label them C and D) will be the points of tangency.

Step 6: Draw $\overleftrightarrow{BC}$ and $\overleftrightarrow{BD}$. These lines are tangent to circle A from a point B outside circle A.

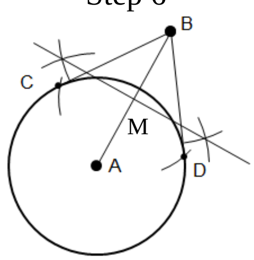
Step 3



Step 5



Step 6



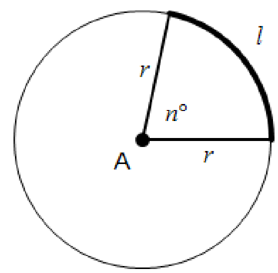
Other helpful construction resources:
http://www.khanacademy.org/math/geometry/circles-topic/v/right-triangles-inscribed-in-circles--proof;
http://www.mathopenref.com/consttangents.html;
http://www.mathsisfun.com/geometry/construct-triangleinscribe.html;
http://www.mathsisfun.com/geometry/construct-trianglecircum.html;
http://www.benjamin-mills.com/maths/Year11/circle-theorems-proof.pdf

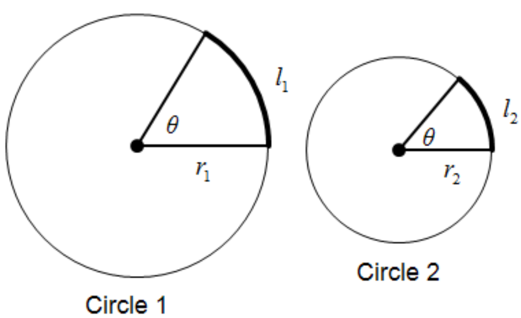
Unit 6 Cluster 2 (G.C.5)
Circles with Coordinates and Without Coordinates

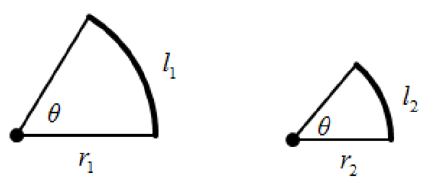
Cluster 2: Finding arc lengths and areas of sectors of circles

- 6.2.1 The length of the arc intercepted by the angle is proportional to the radius
- 6.2.1 The radian measure is the ratio between the intercepted arc and the radius
- 6.2.1 Derive the formula for the area of the sector

Recall that the measure of an arc is the same as the measure of the central angle that intercepts it. The measure of an arc is in degrees, while the arc length is a fraction of the circumference. Thus, the measure of an arc is not the same as the arc length.







Consider the circles and the arcs shown at the left. All circles are similar therefore circle 2 can be dilated so that it is mapped on top of circle 1. The same dilation maps the slice of the small circle to the slice of the large circle. Since corresponding lengths of similar figures are proportional the following relationship exists.

$$\frac{r_1}{r_2} = \frac{l_1}{l_2}$$

Solving this proportion for l_1 gives the following equation.

$$r_1 l_2 = l_1 r_2$$

$$r_1 \frac{l_2}{r_2} = l_1$$

This means that the arc length, l_1, is equal to the radius, r_1, times some number, $\frac{l_2}{r_2}$. We can find that number by looking at how the central angle compares to the entire circle. Given a central angle of $30°$ it is $\frac{30°}{360°}$ or $\frac{1}{12}$ of the entire circle. The length of the arc that is intercepted by the central angle of $30°$ will also be $\frac{1}{12}$ of the circumference. Therefore the length of the arc depends only on the radius.

To generalize the relationship between the arc length and the radius, set up a proportion showing that the central angle compared to the whole circle is proportional to the length of the arc compared to the circumference of the circle.

$$\frac{\theta}{360^{\circ}}=\frac{l}{2\pi r}$$

Find the length of an arc by multiplying the central angle ratio by the circumference of the circle (2πr). In other words solve for *l* (length).

$$l=\frac{\theta}{360^{\circ}}2\pi r$$

This equation can be simplified because 360, 2 and π are all constants.

$$l=\frac{2\pi\theta}{360^{\circ}}r$$

$$l=\frac{\pi\theta}{180^{\circ}}r$$

Compare this formula with the formula we obtained earlier $r_1\frac{l_2}{r_2}=l_1$. The number $\frac{l_2}{r_2}$ is $\frac{\pi\theta}{180^{\circ}}$ where θ is the measure of the central angle in degrees.

Formula for arc length If the central angle of a circle with radius *r* is θ degrees, then the length, *l*, of the arc it intercepts is given by: $l=\frac{\pi\theta}{180^{\circ}}r$.

<u>Example 1: Finding arc length</u>

Find the arc length if the radius of a circle is 5 centimeters and the central angle is 72°. Write the answer in terms of π and give a decimal approximation to nearest thousandth.

Answer:

$l=\frac{\pi\theta}{180^{\circ}}r$ $l=\frac{\pi(72^{\circ})}{180^{\circ}}(5)$	Use the formula for arc length. Substitute 5 in for *r* and 72° in for θ.
$l=\frac{\pi(72^{\circ})}{180^{\circ}}(5)$ $l=\frac{\pi(360^{\circ})}{180^{\circ}}$ $l=2\pi\approx 6.283$	Simplify.
The arc length is 2π centimeters or approximately 6.283 centimeters.	

Example 2: Finding arc length

Find the arc length if the radius of a circle is 7 inches and the central angle is $120°$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

Answer:

$l=\dfrac{\pi\theta}{180°}r$ $l=\dfrac{\pi(120°)}{180°}(7)$	Use the formula for arc length. Substitute 7 in for *r* and $120°$ in for θ.
$l=\dfrac{\pi(120°)}{180°}(7)$ $l=\dfrac{\pi(840°)}{180°}$ $l=\dfrac{14}{3}\pi\approx 14.661$	Simplify.
The arc length is $\dfrac{14}{3}\pi$ inches or approximately 14.661inches.	

Practice Exercises A

1. Find the arc length if the radius of a circle is 10 yards and the central angle is $44°$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

2. Find the arc length if the radius of a circle is 8 meters and the central angle is $99°$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

3. Find the arc length if the radius of a circle is 2 feet and the central angle is $332°$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

4. Find the arc length if the radius of a circle is 3 kilometers and the central angle is $174°$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

5. Find the arc length if the radius of a circle is 9 centimeters and the central angle is $98°$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

6. Find the arc length if the radius of a circle is 6 miles the central angle is $125°$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

Another way to measure angles is with **radians**. The radian measure of a central angle is defined as the ratio of the arc length compared to the radius. If α is the radian measure of a central angle then, $\alpha = \frac{l}{r} = \frac{\frac{\pi\theta}{180^\circ}r}{r} = \frac{\pi\theta}{180^\circ}$, where θ is the measure of the central angle in degrees. To convert any angle in degrees to radian measure multiply the angle in degrees by $\frac{\pi}{180^\circ}$.

Converting Between Radians and Degrees
To convert degrees to radians, multiply the angle by $\frac{\pi \text{ radians}}{180^\circ}$.
To covert radians to degrees, multiply the angle by $\frac{180^\circ}{\pi \text{ radians}}$.

Example 3: Converting from degrees to radians

Convert an angle of 25° to radian measure. Leave your answer in terms of π.

Answer:

$25^\circ \cdot \frac{\pi \text{ radians}}{180^\circ}$ $\frac{25^\circ}{180^\circ} \cdot \pi \text{ radians}$	Multiply the angle by the conversion factor $\frac{\pi \text{ radians}}{180^\circ}$.
$\frac{5}{36}\pi \text{ radians}$	Simplify the fraction.

Example 4: Converting from radians to degrees

Convert an angle of $\frac{\pi}{2}$ radians to degrees.

Answer:

$\frac{\pi}{2} \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}}$ $\frac{180^\circ}{2} \cdot \frac{\pi \text{ radians}}{\pi \text{ radians}}$	Multiply the angle by the conversion factor $\frac{180^\circ}{\pi \text{ radians}}$.
90°	Simplify the fraction.

Practice Exercises B

Find the degree measure of each angle expressed in radians and find the radian measure of each angle expressed in degrees. (Express radian measures in terms of π.)

1. 135°
2. $\frac{2\pi}{3}$
3. -45°
4. $-\frac{5\pi}{4}$
5. 330°
6. $\frac{5\pi}{2}$

The arc length can also be found by using a radian measure for the central angle. When this happens the formula is $l = \theta r$, where θ is in radian measure.

Practice Exercises C
Compare your answers found here to those in Practice Exercises A.

1. Find the arc length if the radius of a circle is 10 yards and the central angle is $\frac{11\pi}{45}$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

2. Find the arc length if the radius of a circle is 8 meters and the central angle is $\frac{11\pi}{20}$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

3. Find the arc length if the radius of a circle is 2 feet and the central angle is $\frac{83\pi}{45}$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

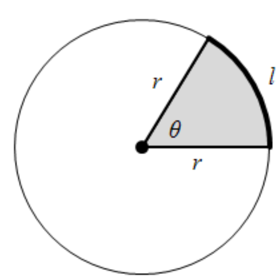
4. Find the arc length if the radius of a circle is 3 kilometers and the central angle is $\frac{29\pi}{30}$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

5. Find the arc length if the radius of a circle is 9 centimeters and the central angle is $\frac{49\pi}{90}$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

6. Find the arc length if the radius of a circle is 6 miles the central angle is $\frac{25\pi}{36}$. Write the answer in terms of π and give a decimal approximation to the nearest thousandth.

Area of a Sector

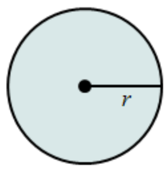
A region of a circle determined by two radii and the arc intercepted by the radii is called a **sector** of the circle (think of a slice of pie). A **sector** is a fraction of a circle, so the ratio of the area of the sector to the area of the entire circle is equal to the measure of the central angle creating the sector to the measure of the entire circle.



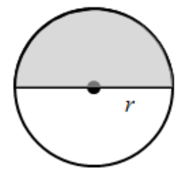
Symbolically this ratio is $\frac{\text{area of a sector}}{\text{area of circle}} = \frac{\text{measure of central angle}}{\text{measure of circle}}$.

Using substitution this becomes, $\frac{\text{area of a sector}}{\pi r^2} = \frac{\theta}{360°}$. Solving for the area of a sector we get: area of a sector $= \frac{\theta}{360°} \cdot \pi r^2$.

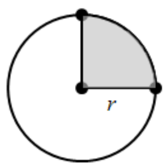
Another way of looking at area of a sector is shown below.



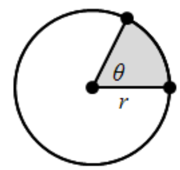
The area of a circle is $A = \pi r^2$.



The area of one-half a circle is $A = \frac{1}{2}\pi r^2$.



The area of one-fourth circle is $A = \frac{1}{4}\pi r^2$.



The area of any fraction of a circle is $A = \frac{\theta}{360}\pi r^2$.

Area of a sector

If the angle θ is in degrees then the area of a sector, *A*, is $A = \frac{\pi\theta}{360} r^2$.

(Rewriting the formula for area of a sector you get $A = \frac{1}{2} \cdot \frac{\pi\theta}{180} r^2$, recall that an angle in radian measure is equal to $\frac{\pi\theta}{180}$ where θ is in degrees.)

If the angle θ is in radian measure then the area of a sector, *A*, is $A = \frac{1}{2}\theta r^2$.

<u>Example 5:</u>

Find the area of a sector with radius 5 cm and central angle of $135°$. Express your answer in terms of π and approximate it to the nearest thousandths.

Answer:

$A = \frac{\pi\theta}{360} r^2$	Use the formula for the area of a sector with the angle in degrees.
$A = \frac{\pi(135°)}{360}(5)^2$ $A = \pi\left(\frac{3}{8}\right)(25)$ $A = \frac{75}{8}\pi$	Substitute known values. $\theta = 135°$ and $r = 5$ Simplify.
The area of the sector is $\frac{75}{8}\pi \approx 29.452$ cm^2.	

Practice Exercises D

Find the area of the sector given the radius and central angle. Express your answer in terms of π and approximate it to the nearest thousandths.

1. A radius of 2 feet and a central angle of 180°.
2. A radius of 5 centimeters and a central angle of 90°.
3. A radius of 4 inches and a central angle of 60°.
4. A radius of 10 inches and a central angle of 120°.
5. A radius of 10 meters and a central angle of 45°.
6. A radius of 7 centimeters and a central angle of $\frac{\pi}{2}$.
7. A radius of 2 millimeters and a central angle of $\frac{5}{6}\pi$.
8. A radius of 6 feet and a central angle of $\frac{5}{4}\pi$.
9. A radius of 3 inches and a central angle of $\frac{3}{2}\pi$.
10. A radius of 6 meters and a central angle of $\frac{5}{3}\pi$.

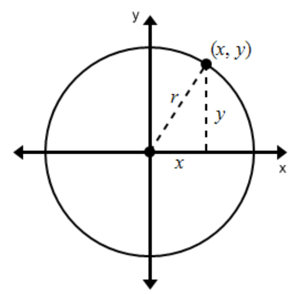
Unit 6 Cluster 3 (G.GPE.1)
Equation of a Circle

Cluster 3: Translating between descriptions and equations for a conic section

6.3.1 Find the equation of a circle given the center and the radius using the Pythagorean Theorem; also complete the square to find the center of the circle given the equation

Deriving the Equation of a Circle Centered at the Origin

A circle is the set of all points equidistant from a given point called the center. In order to derive the equation of a circle, we need to find the distance from the center to any point (x, y) on the circle. This distance is the length of the radius. We can derive the equation of a circle using the distance formula or the Pythagorean Theorem.

Create a right triangle with the radius as the hypotenuse, the length *x* as the horizontal leg, and the length *y* as the vertical leg. Using the Pythagorean Theorem to relate the sides we get: $r^2 = x^2 + y^2$. Where *r* is the radius and the circle is centered at the origin.	

Example 1:

Identify the center and radius of the circle with equation $x^2 + y^2 = 49$.

Answer:

$x^2 + y^2 = 49$ $x^2 + y^2 = 7^2$	Rewrite the equation so that it matches the standard form of the equation of a circle. The square root of 49 is 7, therefore $7^2 = 49$.
The center is at the origin and the radius is 7.	

Example 2:

Write the equation of a circle centered at the origin with a radius of 2.

Answer:

$x^2 + y^2 = r^2$	Use the standard form of the equation of a circle.
$x^2 + y^2 = 2^2$ $x^2 + y^2 = 4$	Substitute known values. $r = 2$ Simplify.

Practice Exercises A

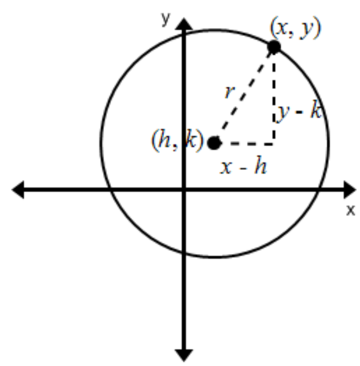
1. Identify the center and the radius for the circle with the equation $x^2 + y^2 = 36$.

2. Write an equation for a circle centered at the origin with a radius of 5.

3. Write an equation for a circle centered at the origin with a diameter of 12.

4. Write an equation for a circles centered at the origin that contains the point $(3,5)$.

Deriving the Equation of a Circle Centered at (h,k)

Find any point (x, y) on the circle. Create a right triangle with the radius as the hypotenuse, the length $x-h$ as the horizontal leg, and the length $y-k$ as the vertical leg. Then using the Pythagorean Theorem to relate the sides we get: $$r^2 = (x-h)^2 + (y-k)^2.$$ Where r is the radius and the circle is centered at the point (h,k).	

Example 3:

Find the center and radius for the circle with equation $(x-3)^2 + (y+2)^2 = 81$.

Answer:

$(x-3)^2 + (y-(-2))^2 = 9^2$ The center is $(3,-2)$ and the radius is 9.	Rewrite the circle's equation so that it matches the standard form of a circle. The square root of 81 is 9 so $9^2 = 81$.

Example 4:
Write the equation for the circle centered at $(-1,5)$ and radius 4.

Answer:

$(x-h)^2+(y-k)^2=r^2$	Start with the standard form of the equation of a circle.
$(x-(-1))^2+(y-5)^2=4^2$	Substitute the known values. $h=-1$, $k=5$, and $r=4$.
$(x+1)^2+(y-5)^2=16$	Simplify.

Example 5:
Write the equation for the circle that has a diameter with endpoints $(-1,5)$ and $(5,-3)$.

Answer:

$\left(\frac{-1+5}{2},\frac{5+(-3)}{2}\right)$ $\left(\frac{4}{2},\frac{2}{2}\right)$ $(2,1)$	Find the center of the circle by finding the midpoint of the diameter.
$r=\sqrt{\left(2-(-1)^2\right)+(1-5)^2}$ $r=\sqrt{(2+1)^2+(-4)^2}$ $r=\sqrt{3^2+16}$ $r=\sqrt{9+16}$ $r=\sqrt{25}$ $r=5$	Find the length of the radius by finding the distance between the center and either of the endpoints of the diameter.
$(x-2)^2+(y-1)^2=5^2$ $(x-2)^2+(y-1)^2=25$	Substitute the known values into the standard form of the equation of a circle centered at (h,k) and simplify. $h=2$, $k=1$, and $r=5$

Practice Exercises B

Given the standard form of a circle determine the center and the radius of each circle.

1. $(x-2)^2+(y-3)^2=16$
2. $(x-1)^2+(y-7)^2=25$
3. $(x+5)^2+(y-6)^2=4$
4. $(x+2)^2+(y-9)^2=36$
5. $(x-10)^2+(y+21)^2=196$
6. $(x+1)^2+(y+3)^2=49$

Write the standard form of a circle with the given characteristics.

7. A circle with radius 10 centered at $(8,-6)$.
8. A circle with radius 5 centered at $(-4,3)$.
9. A circle with diameter endpoints at $(9,2)$ and $(-1,6)$.
10. A circle with diameter endpoints at $(3,4)$ and $(-5,2)$.

Example 6:

Complete the square to find the center and radius of a circle given by the equation $x^2+y^2-6x+2y-6=0$.

Answer:

$x^2-6x+y^2+2y=6$	Collect the *x* terms together, the *y* terms together and move the constant to the other side of the equation.
$\left(x^2-6x+\_\_\_\right)+\left(y^2+2y+\_\_\_\right)=6$ $\left(x^2-6x+\left(\frac{-6}{2}\right)^2\right)+\left(y^2+2y+\left(\frac{2}{2}\right)^2\right)=6+9+1$ $\left(x^2-6x+9\right)+\left(y^2+2y+1\right)=16$	Group the *x* and *y* terms together. Complete the square and simplify.
$(x-3)^2+(y+1)^2=16$	Rewrite each trinomial as a binomial squared.
The center is at $(3,-1)$ and the radius is 4.	

Practice Exercises C

Complete the square to find the center and radius of a circle given by the equation.

1. $x^2 + y^2 - 4x - 6y + 8 = 0$

2. $x^2 + y^2 - 4x + 10y + 20 = 0$

3. $x^2 + y^2 + 6x - 2y - 15 = 0$

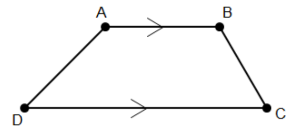
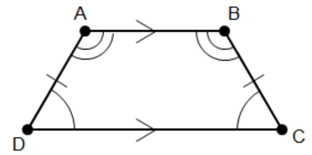
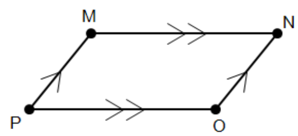
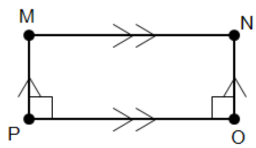
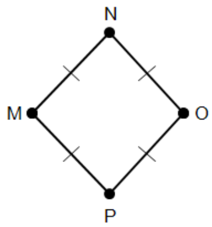
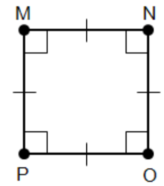
4. $x^2 + y^2 - 6x + 4y + 9 = 0$

Challenge Problem $2x^2 + 2y^2 + 6x - 8y + 12 = 0$

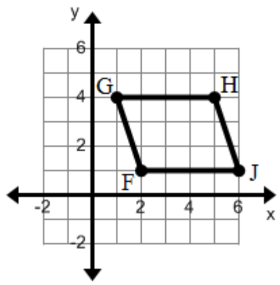
Unit 6 Cluster 4 (G.GPE.4)
Proving Geometrical Theorems Algebraically

Cluster 4: Using coordinates to prove theorems algebraically

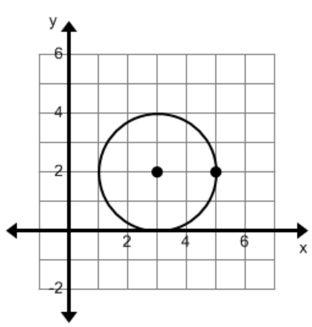
6.4.2 Use coordinates to prove geometric theorems algebraically (include simple proofs involving circles)

A **trapezoid** is a quadrilateral with only one set of parallel sides.	
An **isosceles trapezoid** is a trapezoid with congruent legs and congruent base angles. The diagonals of an isosceles trapezoid are congruent.	
A **parallelogram** is a quadrilateral with opposite sides parallel and congruent. The diagonals of a parallelogram bisect each other.	
A **rectangle** is a special parallelogram with four right angles. The diagonals of a rectangle are congruent.	
A **rhombus** is a special parallelogram with four congruent sides. The diagonals bisect each other and are perpendicular to one another.	
A **square** is a special rectangle and rhombus with four congruent sides. The diagonals are congruent, bisect each other, and are perpendicular to each other.	

Example 1:
Prove *FGHJ* with vertices $F(2,1)$, $G(1,4)$, $H(5,4)$, and $J(6,1)$ is a parallelogram.

	Plot the points on a coordinate plane.
$GF=\sqrt{(2-1)^2+(1-4)^2}=\sqrt{1^2+(-3)^2}=\sqrt{1+9}=\sqrt{10}$ $HJ=\sqrt{(5-6)^2+(4-1)^2}=\sqrt{(-1)^2+3^2}=\sqrt{1+9}=\sqrt{10}$ $FJ=\sqrt{(6-2)^2+(1-1)^2}=\sqrt{4^2+0^2}=\sqrt{16}=4$ $GH=\sqrt{(5-1)^2+(4-4)^2}=\sqrt{4^2+0^2}=\sqrt{16}=4$	Find the distance of $\overline{GF}$, $\overline{HJ}$, $\overline{FJ}$, and $\overline{GH}$.
$m_{GF}=\frac{1-4}{2-1}=\frac{-3}{1}=-3$ $m_{HJ}=\frac{4-1}{5-6}=\frac{3}{-1}=-3$ $m_{FJ}=\frac{1-1}{6-2}=\frac{0}{4}=0$ $m_{GH}=\frac{4-4}{5-1}=\frac{0}{4}=0$	Find the slopes of $\overline{GF}$, $\overline{HJ}$, $\overline{FJ}$, and $\overline{GH}$.
Since opposite sides are congruent and parallel the quadrilateral *FGHJ* is a parallelogram.	

Example 2:
Prove that the point (5,2) is on the circle with center (3,2) and radius 2.

	Create the circle on a coordinate plane.

$r=\sqrt{(5-3)^2+(2-2)^2}$ $r=\sqrt{2^2+0^2}$ $r=\sqrt{4}$ $r=2$	Use the distance formula to show that the radius is 2. Use the center as (x_1, y_1) and $(5,2)$ as (x_2, y_2).
The radius of the circle is 2 therefore the point must lie on the circle.	

Practice Exercises A

1. Prove that quadrilateral EFGH is an isosceles trapezoid given the following vertices: $E(-3,2)$, $F(2,2)$, $G(3,-2)$, and $H(-4,-2)$.

2. Prove that quadrilateral ABCD is a parallelogram given vertices: $A(2,3)$, $B(3,6)$, $C(6,8)$ and $D(5,5)$.

3. Prove that ABCD is a parallelogram given vertices: $A(-3,2)$, $B(0,4)$, $C(1,8)$ and $D(-2,6)$ using distance and slope.

4. Prove that PQRS is not a rectangle given vertices: $P(0,2)$, $Q(2,5)$, $R(5,5)$, and $S(4,2)$.

5. Prove that ABCD is a parallelogram given vertices: $A(2,4)$, $B(6,4)$, $C(5,0)$, and $D(1,0)$.

6. Prove that ABCD is a rhombus given vertices: $A(-1,2)$, $B(0,6)$, $C(4,7)$, and $D(3,3)$.

7. Prove that the point $(3,\sqrt{3})$ lies on the circle with radius 2 and center $(2,0)$.

8. Prove that the point $(2,-5)$ lies on the circle with radius 2 and center $(2,-3)$.

9. Given a circle with center $(-2,3)$ determine whether or not the points $(-4,-1)$ and $(3,5)$ are on the same circle.

10. Given a circle with center at the origin determine whether or not the points $(1,\sqrt{3})$ and $(0,2)$ lie on the same circle.

11. Prove that the line containing the points $(-5,3)$ and $(3,3)$ is tangent to the circle with equation $(x-1)^2+(y+1)^2=100$.

12. Prove that the line containing the points $(0,8)$ and $(4,11)$ is tangent to the circle with equation $(x-2)^2+(y+3)^2=100$.

Other helpful resources:
http://www.regentsprep.org/regents/math/geometry/GCG4/CoordinatepRACTICE.htm
http://staff.tamhigh.org/erlin/math_content/Geometry/PolyQuad/CoordinateGeometryNotes.pdf
http://mtprojectmath.com/Resources/Coordinate%20Geometry%20Proofs.pdf

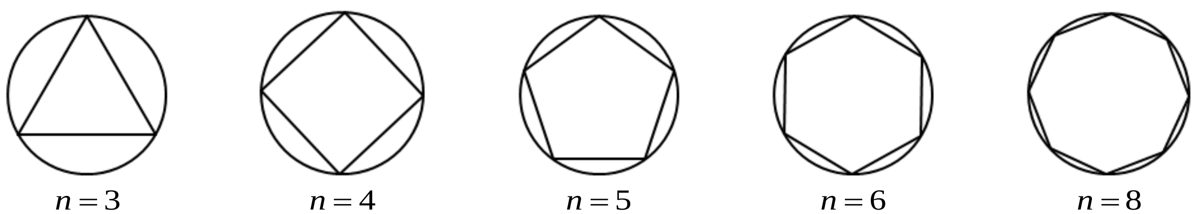
Unit 6 Cluster 5 (G.GMD.1 and G.GMD.3, Honors G.GMD.2) Formulas and Volume

Cluster 5: Explaining and using volume formulas

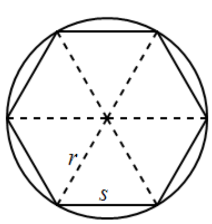
- 6.5.1 Informal arguments for circumference of circle, volume of a cylinder, pyramid, and cone (use dissection arguments, Cavalieri's principle) (use the relationship of scale factor *k*, where *k* is for a single length, k^2 is area, k^3 is volume)
- H.6.1 Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
- 6.5.2 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems

Formula for Circumference

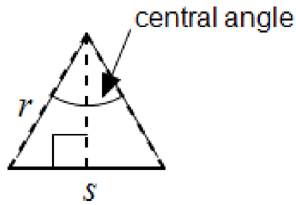
The circumference of a circle is $2\pi r$. The perimeter of a regular polygon inscribed in a circle gives an estimate of the circumference of a circle. By increasing the number of sides, *n*, of the regular polygon we increase the accuracy of the approximation.



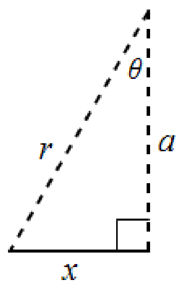
If we had a regular polygon with an infinite number of sides, then we would have the exact value of the circumference of a circle. To prove this numerically, we need a formula for the perimeter of a regular polygon with *n* sides. The perimeter of a regular polygon is found by multiplying the number of sides by the side length. Symbolically that is $P = ns$. If we can find the value for any side length, *s*, then we can find the perimeter for any regular polygon with *n* sides.



Given a regular polygon with side length *s* and radius *r* we can divide the polygon into *n* congruent isosceles triangles.



Each triangle has a hypotenuse of length *r* and a base of length *s*. If we construct a perpendicular bisector from the central angle to the base, we can use trigonometric ratios to find the length of *s*.



Since the perpendicular bisector, *a*, divides the side of length, *s*, in half $s = 2x$. Thus, we need to find the value *x*. The opposite side is *x* and the hypotenuse is *r* therefore $\sin\theta = \frac{x}{r}$. By multiplying each side of the equation by *r* we can isolate *x* and we get $x = r\sin\theta$. Since $s = 2x$, $s = 2r\sin\theta$. Substituting this value into our perimeter equation yields $P = ns = n \cdot 2r\sin\theta$.

The central angle is found by dividing $360°$ by the number of sides, *n*. The angle θ is half of the central angle. Thus, $\theta = \frac{360}{2n}$.

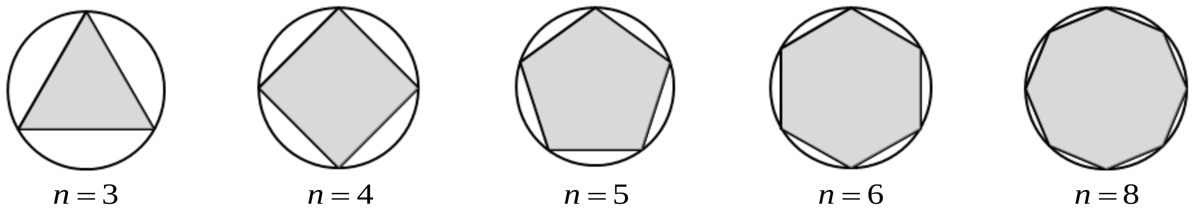
We can use this equation to show that as *n* increases the perimeter approaches $2\pi r \approx 6.283r$. The following table demonstrates this relationship.

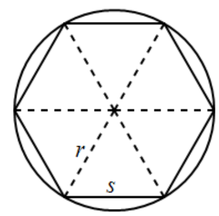
n	$\theta = \frac{360°}{2n}$	**Perimeter Expression** $P = n \cdot 2r\sin\theta$	**Approximation**
3	$\theta = \frac{360°}{2(3)} = 60°$	$P = 3 \cdot 2r\sin(60°)$	$5.196r$
4	$\theta = \frac{360°}{2(4)} = 45°$	$P = 4 \cdot 2r\sin(45°)$	$5.656r$
5	$\theta = \frac{360°}{2(5)} = 36°$	$P = 5 \cdot 2r\sin(36°)$	$5.878r$
6	$\theta = \frac{360°}{2(6)} = 30°$	$P = 6 \cdot 2r\sin(30°)$	$6.000r$
10	$\theta = \frac{360°}{2(10)} = 18°$	$P = 10 \cdot 2r\sin(18°)$	$6.180r$
100	$\theta = \frac{360°}{2(100)} = 1.8°$	$P = 100 \cdot 2r\sin(1.8°)$	$6.282r$
1000	$\theta = \frac{360°}{2(1000)} = 0.18°$	$P = 1000 \cdot 2r\sin(0.18°)$	$6.283r$

A regular polygon with 1000 sides is accurate to four decimal places for the approximation of $2\pi r$. With an infinite number of sides, the regular polygon would essentially be a circle and the perimeter would equal $2\pi r$. Therefore, the circumference of a circle is $2\pi r$.

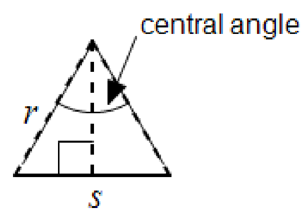
Area of a Circle

A similar process can be used to show that the area of a circle is πr^2. The area of a regular polygon inscribed in a circle gives an estimate of the area of a circle. By increasing the number of sides, *n*, of the regular polygon we increase the accuracy of the approximation of the area of a circle.

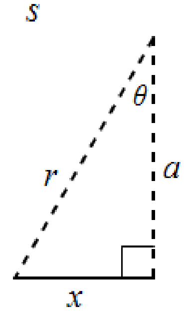




Because a regular polygon with *n* sides of side length *s* can be divided into *n* isosceles triangles, the area of the regular polygon can be found by multiplying the area of one isosceles triangle by the number of triangles formed. Symbolically, $A = \text{area of triangle} \cdot n$ or $A = \left(\frac{1}{2}bh\right)n$.



The base is length *s* and we know that $s = 2r\sin\theta$. We need to find the height, which is the altitude of the triangle and is sometimes called the apothem. This can be done by using trigonometric ratios.



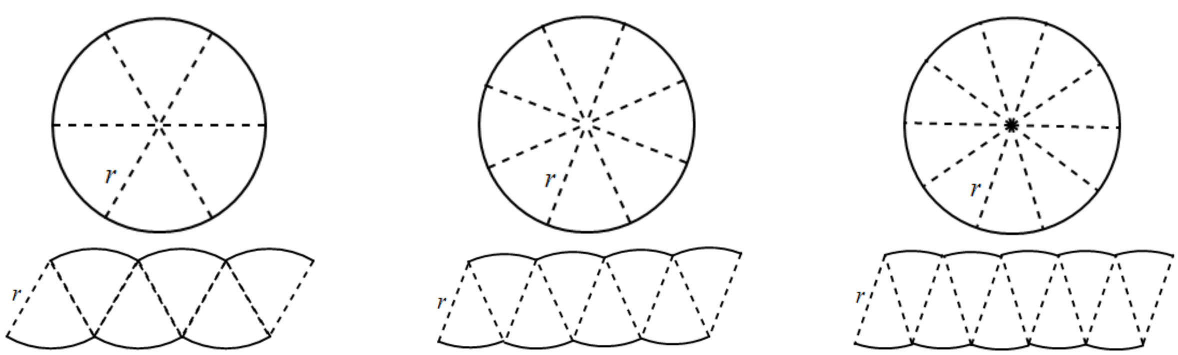
The altitude is adjacent to the angle and the radius is the hypotenuse. So, $\cos\theta = \frac{a}{r}$. By multiplying each side by *r* we can isolate *a* and we get $a = r\cos\theta$.

By substituting these values into our area formula we get $A=\left(\frac{1}{2}bh\right)n=\frac{1}{2}(2r\sin\theta)(r\cos\theta)n$ simplifying it we get $A=r^2 n\sin\theta\cos\theta$. We can use this equation to show that as *n* increases the area approaches $\pi r^2 \approx 3.142r^2$. The following table demonstrates this relationship.

n	$\theta=\frac{360°}{2n}$	**Area Expression** $A=r^2 n\sin\theta\cos\theta$	**Approximation**
3	$\theta=\frac{360°}{2(3)}=60°$	$A=r^2(3)\sin(60°)\cos(60°)$	$1.299r^2$
4	$\theta=\frac{360°}{2(4)}=45°$	$A=r^2(4)\sin(45°)\cos(45°)$	$2.000r^2$
5	$\theta=\frac{360°}{2(5)}=36°$	$A=r^2(5)\sin(36°)\cos(36°)$	$2.378r^2$
6	$\theta=\frac{360°}{2(6)}=30°$	$A=r^2(6)\sin(30°)\cos(30°)$	$2.598r^2$
10	$\theta=\frac{360°}{2(10)}=18°$	$A=r^2(10)\sin(18°)\cos(18°)$	$2.939r^2$
100	$\theta=\frac{360°}{2(100)}=1.8°$	$A=r^2(100)\sin(1.8°)\cos(1.8°)$	$3.140r^2$
1000	$\theta=\frac{360°}{2(1000)}=0.18°$	$A=r^2(1000)\sin(0.18°)\cos(0.18°)$	$3.142r^2$

A regular polygon with 1000 sides is accurate to four decimal places for the approximation of πr^2. With an infinite number of sides the regular polygon would essentially be a circle and the area would equal to πr^2. Therefore, the area of a circle is πr^2.

A simpler way to look at this is by slicing the circle into infinitely many slices and arranging those slices so that they form a parallelogram. The smaller the slice, the more linear the intercepted arc becomes and the more it looks like a parallelogram.



$n=6$ $n=8$ $n=10$

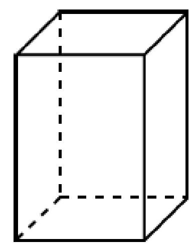
The base of the parallelogram is half of the circumference. Therefore, $b=\frac{1}{2}(2\pi r)=\pi r$. The height is the radius. The area of a parallelogram can be found by multiplying the base times the height. Symbolically, $A=bh=(\pi r)r=\pi r^2$.

Volume

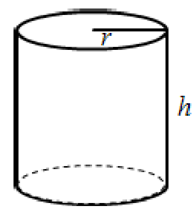
VOCABULARY

The **volume** of a three dimensional figure is the space that it occupies.

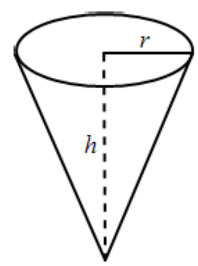
A **prism** is a three-dimensional figure with two congruent and parallel faces that are called **bases**.



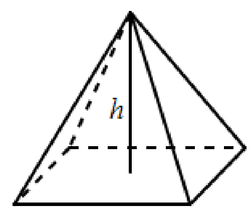
A **cylinder** is a three-dimensional figure with parallel bases that are circles.



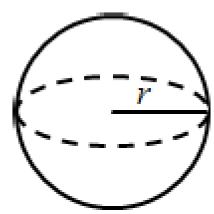
A **cone** is a three-dimensional figure that has a circle base and a vertex that is not in the same plane as the base. The height of the cone is the perpendicular distance between the vertex and the base.

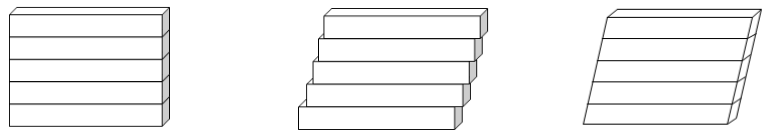


A **pyramid** is a three-dimensional figure with a polygon as its base and triangles as its lateral faces. The triangles meet at a common vertex.



A **sphere** is the set of all points in space that are the same distance from the center point.





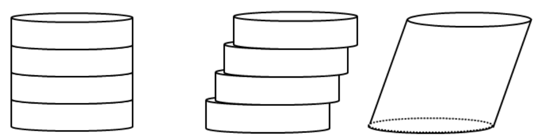
The three figures above have the same volume. Each figure has the same number of levels and each level has the same volume. This illustrates Cavalieri's Principle.

Cavalieri's Principle
If two space figures have the same height and the same cross-sectional area at every level, then they have the same volume.

The volume of any prism or cylinder can be found by multiplying the area of the base times the height, $V = Bh$.

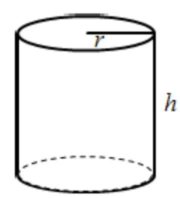
Volume of a Cylinder

Extending the idea of Cavalieri's principle, the volume of a cylinder is the area of its base, a circle, times the perpendicular height. Symbolically, $V = Bh = \pi r^2 h$.



Formula for the Volume of a Cylinder

The volume of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.



<u>Example 1: Volume of a cylinder</u>
The radius of a circular container is 4 inches and the height is 10 inches. Find the volume of the container.

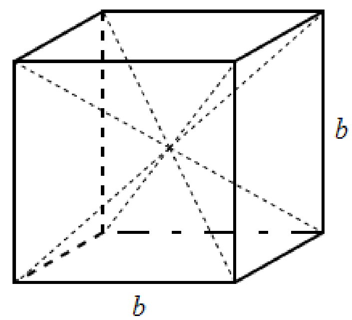
Answer:

$V = \pi r^2 h$	Use the formula for the volume of a cylinder.
$V = \pi(4)^2(10)$	Substitute in known values. $r = 4$ and $h = 10$
$V = \pi(16)(10)$ $V = 160\pi \approx 502.655$	Simplify.
The volume is approximately 502.655 in^3.	Volume is measured in cubic units.

Volume of a Pyramid

The volume of a cube can help us find the volume of a pyramid. Since a cube has 6 faces that are all squares, the volume of a cube is $V = Bh = b^2 \cdot b = b^3$, where b is the length of the side.

Inside the cube, place 6 pyramids that have a face of the cube as a base and share a vertex at the center of the cube. The six pyramids are equally sized square pyramids.



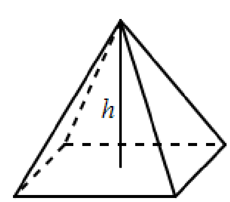
Using the formula for the volume of a cube, we can derive the volume of one of the pyramids.

$V = b^3$ $V = Bh$ $V = b^2 \cdot b$	Begin with the volume of a cube. Rewrite it so that it is in the form $V = Bh$.
$V = \frac{1}{6}b^2 \cdot b$	There are 6 equally sized pyramids within the cube. The volume of one pyramid will be equal to one-sixth the volume of the cube.
$V = \frac{1}{6}b^2 \cdot b$ $V = \frac{1}{6}b^2(2h)$	Rewrite the formula using the height of the pyramids. Use $b = 2h$ because the height of the cube is equivalent to the height of two pyramids.
$V = \frac{2}{6}b^2h$ $V = \frac{1}{3}b^2h$	Simplify.

In general, the volume of a pyramid is $V = \frac{1}{3}b^2h = \frac{1}{3}Bh$. In other words, the volume of a pyramid is the area of its base, regardless of whether or not it is a square, times the height.

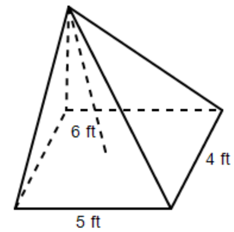
Formula for the Volume of a Pyramid

For any pyramid with area of base, B, and height h, the volume is $V = \frac{1}{3}Bh$.



Example 2: Volume of a Pyramid

Find the volume of the pyramid with a rectangular base that is 4 ft by 5 ft and height of 6 ft.

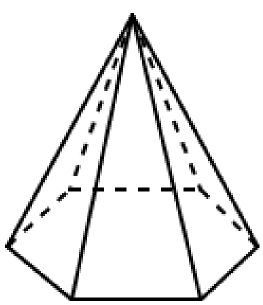


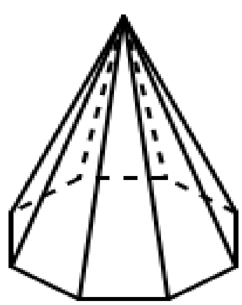
Answer:

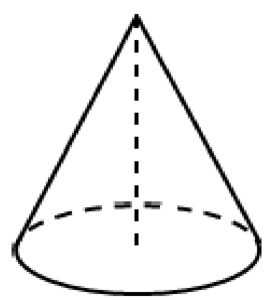
$V = \frac{1}{3}Bh$	Use the formula for the volume of a pyramid.
$V = \frac{1}{3}(20)(6)$	The base is a rectangle so the area of the base is $A = lw = 4 \cdot 5 = 20$. The height is 6.
$V = \frac{1}{3}(120)$ $V = 40$	Simplify.
The volume of the pyramid is 40 ft^3.	Volume is measured in cubic units.

Volume of a Cone

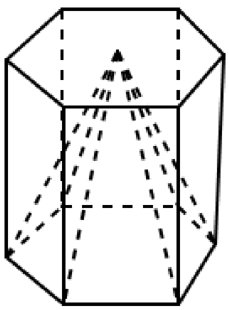
The relationship between the volume of a cone and the volume of a cylinder is similar to the relationship between the volume of a pyramid and the volume of a prism. We already know that the volume of a pyramid is one-third the volume of a prism. We want to show that the volume of a cone is also one-third the volume of a cylinder. To do this, we are going to look at a pyramid with a base that has *n* sides. If we increase the number of sides until there are an infinite number of sides, the pyramid will have a circular base and be a cone.


6 sided base

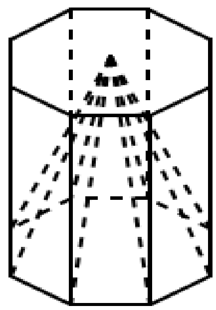

8 sided base


Inifinite sided base

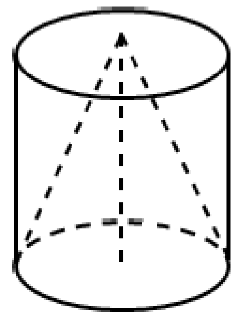
When we increase the number of sides of the base of pyramid, we also change the shape of the prism that holds the pyramid. The prism that contains the pyramid with an infinite number of sides is a cylinder.



6 sided base



8 sided base

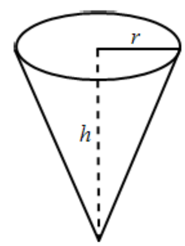


Infinite sided base

Since a cone is essentially a pyramid with an infinite number of sides and a cylinder is a prism with a base that has an infinite number of sides, the volume of a cone is one-third the volume of a cylinder.

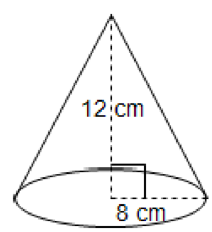
Formula for the Volume of a Cone

For any cone with radius, *r*, and height *h*, the volume is $V = \frac{1}{3}\pi r^2 h$.



Example 3: Volume of a cone

Find the volume of the cone with radius 8 cm and height 12 cm.



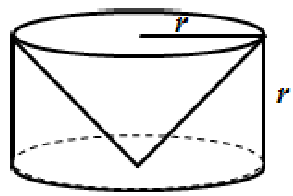
Answer:

$V = \frac{1}{3}\pi r^2 h$	Use the formula for the volume of a cone.
$V = \frac{1}{3}\pi (8)^2 (12)$	Substitute in known values. $r = 8$ and $h = 12$
$V = \frac{1}{3}\pi (64)(12)$ $V = \frac{1}{3}\pi (768)$ $V = 256\pi \approx 804.248$	Simplify.
The volume of the cone is approximately 804.248 cm^3.	Volume is measured in cubic units.

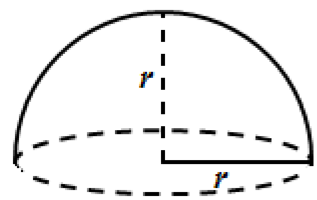
Volume of a Sphere

In order to derive the formula for the volume of a sphere, we need to review Cavalieri's Principle. That is, that if two figures have the same height and same cross-sectional area at every level, then they have the same volume. We will also use the volume of a cone $V=\frac{1}{3}\pi r^2 h$ and the volume of a cylinder $V=\pi r^2 h$.

Consider the cylinder and hemisphere given below, both with a radius of *r* and height *r*. From the cylinder, a cone is cut out that shares the same base as the cylinder and also has a radius of *r* and a height of *r*. We are going to prove that this "cone-less cylinder", or the part of the cylinder that remains after the cone has been removed, has the same volume as the hemisphere. Once we have proven that they have the same volume, we can merely double the formula for the "cone-less cylinder" to obtain the volume for a sphere.



Cylinder

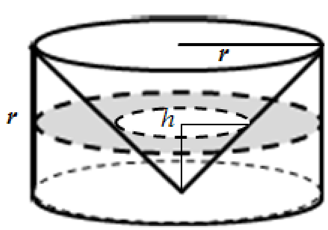


Hemisphere

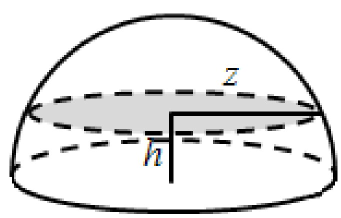
We start calculating the volume of the "cone-less cylinder" by subtracting the volume of the cone from the volume of the cylinder. Since the height is *r*, the volume of this cone would be $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi r^2\cdot r=\frac{1}{3}\pi r^3$ and the volume of the cylinder would be $V=\pi r^2 h=\pi r^2\cdot r=\pi r^3$.

Upon subtracting their volumes we get $V=\pi r^3-\frac{1}{3}\pi r^3=\frac{2}{3}\pi r^3$. So the volume of the "cone-less cylinder" is $V=\frac{2}{3}\pi r^3$

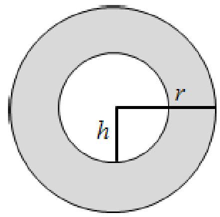
In order to use Cavalieri's Principle, we are going to create circular cross sections of the figures by slicing them at height *h* and comparing their areas. The cross section of the cylinder and the hemisphere are shown below.



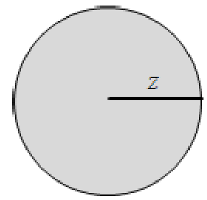
Cylinder with cone inside



Hemisphere



Cross sectional slice of cylinder with cone

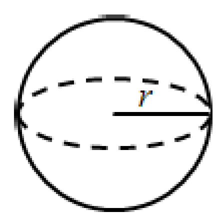


Cross sectional slice of hemisphere

If we can prove that these cross sections have the same area, then by Cavalieri's Principle the volume of the hemisphere and the volume of the "cone-less" cylinder are the same. Start by finding the area of the cross section of the hemisphere, or the disk shaped cross section. We need to find its radius, *z*. Since the cross section is *h* units above the base of the hemisphere, we can use the Pythagorean Theorem and the radius of the hemisphere's base to calculate the cross sectional radius *z* to be $z=\sqrt{r^2-h^2}$. Using this radius we can now calculate the area of the cross section. $A=\pi z^2=\pi\left(\sqrt{r^2-h^2}\right)^2=\pi(r^2-h^2)$. To find the area of the cross-section of the "cone-less cylinder", or the washer shaped cross section, we need to subtract the small circle from the large circle. Since the radius of the cone at any height *h* is proportional to the height with a 1:1 ratio, the large circle (cross-section of the cylinder) has a radius of *r* while the small circle (cross-section of the cone) has a radius of *h*. Thus the area of the large circle is πr^2 and the area of the small circle is πh^2. Their difference is $\pi r^2-\pi h^2=\pi(r^2-h^2)$. Therefore, the area of the cross section from the hemisphere is the same as the area of the cross section from the "cone-less" cylinder. Cavalieri's Principle states that if we sum up all of our equal slices or cross sectional areas then we obtain equivalent volumes for the hemisphere and "cone-less cylinder." Recall that the volume of the cone-less cylinder is $V=\frac{2}{3}\pi r^3$. We can then infer that the volume of the hemisphere is $V=\frac{2}{3}\pi r^3$ and the that volume of the sphere would be twice the volume of the hemisphere or $V=2\cdot\frac{2}{3}\pi r^3=\frac{4}{3}\pi r^3$.

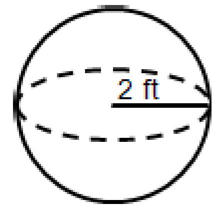
Formula for the Volume of a Sphere

For any sphere with radius *r* the volume is $V=\frac{4}{3}\pi r^3$.



Example 4: Volume of a sphere

Find the volume of a sphere with radius of 2 ft.



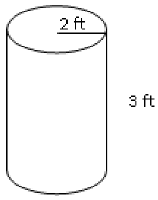
Answer:

$V = \frac{4}{3}\pi r^3$	Use the formula for the volume of a sphere.
$V = \frac{4}{3}\pi (2)^3$	Substitute in known values. $r = 2$
$V = \frac{4}{3}\pi (8)$ $V = \frac{32}{3}\pi \approx 33.5210$	Simplify.
The volume of the sphere is approximately 33.5210 ft^3.	Volume is measured in cubic units.

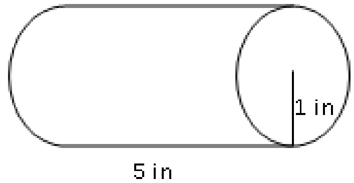
Practice Exercises A

Find the volume of the following.

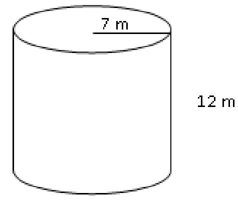
1.



2.



3.



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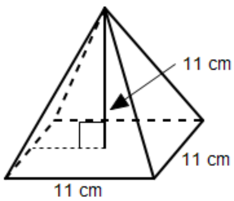
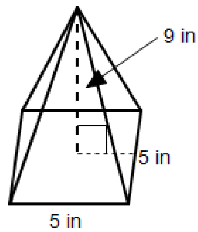


5.



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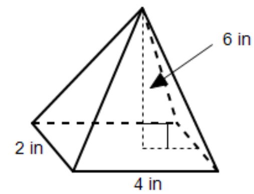

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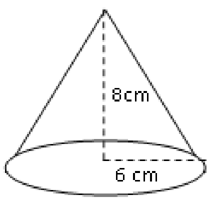

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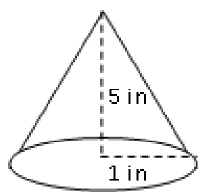
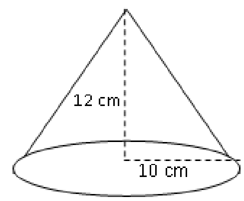
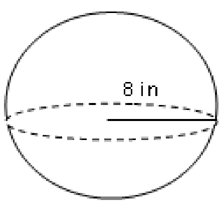


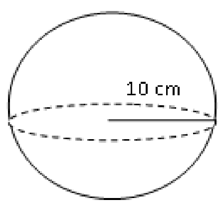
9.



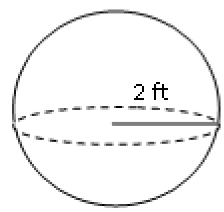
10.



11.



12.



13. Find the volume of a pool if the diameter is 20 feet and the depth is 6 feet.

14. A fish tank is in the shape of a cylinder. The tank is 22 feet deep and 40 feet in diameter. Find the volume of the tank.

15. Popcorn is served in a conical container that has a radius of 3 inches and a height of 6 inches. What is the volume of the small container?

16. A tank is in the shape of a pyramid. The base of the tank is a rectangle with length 3 feet and width 7 feet. It is 9 feet high. Find the volume of the tank.

17. Find the volume of a pyramid if the base is a square with side 6 feet and the height is 4 feet.

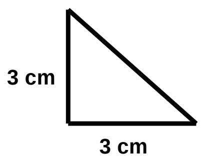
18. How much ice cream is needed to fill a conical shaped sugar cone that is 4 inches deep and 6 inches in diameter?

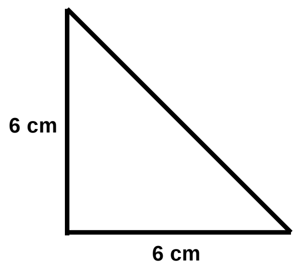
19. A soccer ball has a radius of 4.3 inches. What is the volume of soccer ball?

20. A softball has a diameter of 9.6 cm. What is its volume?

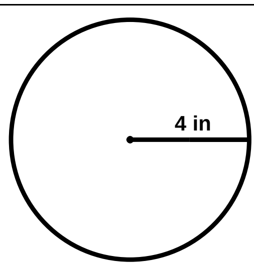
Relating Scale Factor, Length, Area, and Volume

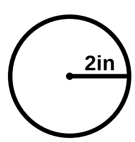
The drawings below are dilations of one another. The length and area of each figure are compared.

<table>
<tr><th>Original Drawing</th><th>Dilated Drawing</th><th>Length Scale Factor $\frac{\text{Dilated Length}}{\text{Original Length}} = k$</th><th>Area Scale Factor $\frac{\text{Dilated Area}}{\text{Original Area}} = k^2$</th></tr>
<tr><td>

$A = \frac{1}{2} \cdot 3 \cdot 3 = 4.5 \text{ cm}^2$</td><td>

$A = \frac{1}{2} \cdot 6 \cdot 6 = 18 \text{ cm}^2$</td><td>$\frac{6}{3} = 2$</td><td>$\frac{18}{4.5} = 4 = 2^2$</td></tr>
<tr><td colspan="4">By comparing the side of the dilated triangle to the corresponding side of the original triangle, we see that the ratio is 2. Thus, the scale fator $k = 2$. If we compare the area of the dilated triangle to the area of the original triangle, the ratio is 4. If we write this in terms of k we get $4 = 2^2$ or k^2.</td></tr>
</table>

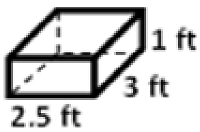
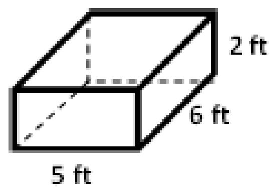
<table>
<tr><th>Original Drawing</th><th>Dilated Drawing</th><th>Length Scale Factor $\frac{\text{Dilated Length}}{\text{Original Length}} = k$</th><th>Area Scale Factor $\frac{\text{Dilated Area}}{\text{Original Area}} = k^2$</th></tr>
<tr><td>

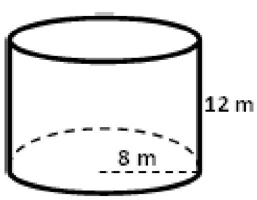
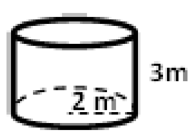
$A = \pi \cdot 6^2 = 36\pi \text{ in}^2$</td><td>

$A = \pi \cdot 2^2 = 4\pi \text{ in}^2$</td><td>$\frac{2}{6} = \frac{1}{3}$</td><td>$\frac{4\pi}{36\pi} = \frac{1}{9} = \left(\frac{1}{3}\right)^2$</td></tr>
<tr><td colspan="4">By comparing the radius of the dilated circle to the radius of the original figure, we see that the ratio is $\frac{1}{3}$. Thus, the scale factor $k = \frac{1}{3}$. If we compare the area of the dilated circle to the area of the original circle, the ratio is $\frac{1}{9}$. If we write this in terms of k we get $\frac{1}{9} = \left(\frac{1}{3}\right)^2$ or k^2.</td></tr>
</table>

Therefore, if a figure is dilated by a scale factor of k, its area is k^2 times the area of the original figure.

The three dimensional figures below are dilations of one another, the length and the volume scale factors of each figure are compared.

Original Drawing	Dilated Drawing	Length Scale Factor $\frac{\text{Dilated Length}}{\text{Original Length}} = k$	Volume Scale Factor $\frac{\text{Dilated Volume}}{\text{Original Volume}} = k^3$
 $V = 2.5 \cdot 3 \cdot 1 = 7.5 \text{ ft}^3$	 $V = 5 \cdot 6 \cdot 2 = 60 \text{ ft}^3$	$\frac{2.5}{5} = 2$	$\frac{60}{7.5} = 8 = 2^3$
By comparing one side of the dilated prism to the corresponding side of the original prism, we find that the ratio is 2. Thus, the scale factor $k = 2$. If we compare the volume of the dilated prism to the volume of the original prism, then the ratio is 8. If we write this in terms of *k*, we $8 = 2^3$ get or k^3.			

Original Drawing	Dilated Drawing	Length Scale Factor $\frac{\text{Dilated Length}}{\text{Original Length}} = k$	Volume Scale Factor $\frac{\text{Dilated Volume}}{\text{Original Volume}} = k^3$
 $V = \pi \cdot 8^2 \cdot 12 = 768\pi \text{ m}^3$	 $V = \pi \cdot 2^2 \cdot 3 = 12\pi \text{ m}^3$	$\frac{3}{12} = \frac{1}{4}$	$\frac{12\pi}{768\pi} = \frac{1}{64} = \left(\frac{1}{4}\right)^3$
By comparing the radius of the dilated cylinder to the radius of the original cylinder, we find that the ratio is $\frac{1}{4}$. Thus, the scale factor $k = \frac{1}{4}$. If we compare the volume of the dilated cylinder to the volume of the original cylinder, then the ratio is $\frac{1}{64}$. If we write this in terms of *k* we get $\frac{1}{64} = \left(\frac{1}{4}\right)^3$ or k^3.			

Therefore, if a figure is dilated by a scale factor of *k*, its volume is k^3 times the volume of the original figure.

Example 5:

The volume of a cylinder is $V = \pi r^2 h$. If the original cylinder has radius 4 inches and height 10 inches how will the volume compare to a cylinder with the same radius but double the height?

Answer:

$V = \pi r^2 h$	Find the volume of each cylinder.
$V = \pi r^2 h$ $V = \pi(4)^2(10)$ $V = \pi(16)(10)$ $V = 160\pi \approx 502.655\,\text{in}^3$	For the original cylinder $r = 4$ and $h = 10$.
$V = \pi r^2 h$ $V = \pi(4)^2(20)$ $V = \pi(16)(20)$ $V = 320\pi \approx 1005.310\,\text{in}^3$	For the cylinder with double the height $r = 4$ and $h = 20$.
$k = \dfrac{\text{volume of original}}{\text{volume of double height}} = \dfrac{160\pi}{320\pi} = \dfrac{1}{2}$	Compare the volumes.
The volume of the original cylinder is half the volume of the cylinder with double the height. The dilation was applied to only one length of the cylinder so it doubled the volume. If the dilation had been applied to the radius and the height, then the volume of the cylinder would have been one-eighth of the cylinder that was double the radius and double the height.	

Example 6:

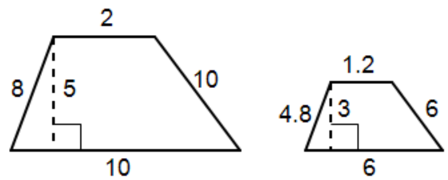
If the original cylinder has radius 4 inches and height 10 inches how will the volume compare to a cylinder with the same height but double the radius?

Answer:

$V = \pi r^2 h$	Find the volume of each cylinder.
$V = \pi r^2 h$ $V = \pi(4)^2(10)$ $V = \pi(16)(10)$ $V = 160\pi \approx 502.655\,\text{in}^3$	For the original cylinder $r = 4$ and $h = 10$.
$V = \pi r^2 h$ $V = \pi(8)^2(10)$ $V = \pi(64)(10)$ $V = 640\pi \approx 2010.619\,\text{in}^3$	For the cylinder with double the radius $r = 8$ and $h = 10$.
$k = \dfrac{\text{volume of original}}{\text{volume of double height}} = \dfrac{160\pi}{640\pi} = \dfrac{1}{4}$	Compare the volumes.
The volume of the original cylinder is one-fourth the volume of the cylinder with double the radius.	

Practice Exercises B

Use the trapezoids to answer questions 1–3.

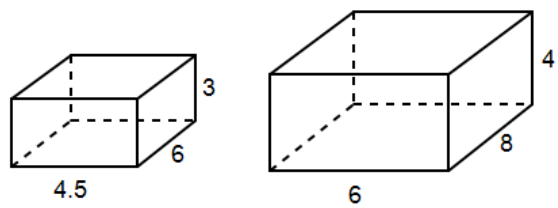


1. What is the ratio of the perimeter of the larger trapezoid to the perimeter of the smaller trapezoid?

2. What is the ratio of the area of the larger trapezoid to the area of the smaller trapezoid? $A=\frac{1}{2}(b_1+b_2)h$

3. What is the scale factor k?

Use the prisms to answer questions 4–6.



4. What is the ratio of the surface area of the smaller prism to the surface area of the larger prism? $S=2lw+2lh+2wh$

5. What is the ratio of the volume of the smaller prism to the volume of the larger prism?

6. What is the scale factor k?

7. You have a circular garden with an area of 32 square feet. If you increase the radius by a scale factor of 5, what is the area of the new garden?

8. Jan made an enlargement of an old photograph. If the ratio of the dimensions of the photograph to the enlargement is 1:3, what will be the ratio of the area of the original photograph to the area of the enlargement?

9. You and your friend are both bringing cylindrical thermoses of water on a camping trip. Your thermos is twice as big as his in all dimensions. How much more water will your thermos carry than your friend's? If your friend's thermos has a diameter of 10 cm and a height of 18 cm, what are the dimensions of your thermos?

10. A cylinder has radius 3 cm and height 5 cm. How does the volume of the cylinder change if both the radius and the height are doubled?

11. A pyramid has a square base with length 4 ft and a height of 7 ft. How does the volume of the pyramid change if the base stays the same and the height is doubled?

12. A pyramid has a square base with length 4 ft and a height of 7 ft. How does the volume of the pyramid change if the height stays the same and the side length of the base is doubled?

13. A movie theater sells a small cone of popcorn for $2. A medium cone of popcorn is sold for $4 and comes in a similar container but it is twice as tall as the small container. Which popcorn size gives you more for your money? Explain your answer.

14. Another movie theater sells a small cone of popcorn for $2. A medium cone of popcorn is sold for $4 and comes in a similar container as the small but the radius is twice the length of the small container. Which popcorn size gives you more for your money? Explain your answer.

15. How much does the volume of a sphere increase if the radius is doubled? Tripled?